University of Alberta

Fluctuations and Non Local Transport in Laser Plasmas

By

Jason F. Myatt ©

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To

my Mother

and

Father.

Abstract

This work is concerned with the theoretical description of thermal fluctuations in classical plasmas that are homogeneous or have background temperature gradients. Attention is focussed on the particularly important mode, the ion acoustic wave. Experimentalists utilize Thomson scattering from ion acoustic fluctuations to extract information such as drift velocities and temperatures. Collisionless theories are reasonably successful in describing Thomson scattering experiments and are usually adopted for simplicity. Including the effects of collisions into the theory will extend the usefulness of Thomson scattering as a diagnostic tool.

Our understanding of collisional effects (e-e), (e-i) and (i-i) on ion acoustic waves has been advanced by an analytical treatment of the electron kinetic equation. The results of this theory have led to a complete description of ion acoustic dispersion and damping over the full range of particle collisionality. The non local electron heat conductivity associated with these waves has also been investigated and the possibility of observing such non local transport effects from Thomson scattering measurements has been examined.

Preface

The work presented in this dissertation was carried out at the Department of Physics at the University of Alberta, between May 1994 and August 1997. The work was done under the supervision of Wojciech Rozmus, and in colaboration with Valery Bychenkov and Vladimir Tikhonchuk. Unless otherwise stated, the work is original and has not previously been submitted for a degree at this or any other university. The material presented is based on the following series of papers:

J. Myatt, W. Rozmus, V. Yu. Bychenkov, and V. T. Tikhonchuk. *Thomson scattering from ion acoustic fluctuations in laser plasmas.* Submitted to Phys. Rev. E.

J. Myatt, W. Rozmus, V. Yu. Bychenkov, and V. T. Tikhonchuk *Comments on Thomson scattering from laser plasmas.* Comments on Plasma Physics and Controlled Thermonuclear Fusion. **17** 331-339 (1996).

V. Yu. Bychenkov, J. Myatt, W. Rozmus and V. T. Tikhonchuk. *Kinetic theory of ion acoustic waves in a plasma with collisional electrons.* Phys. Rev. E. **52** 6759-6776 (1995).

V. Yu. Bychenkov, J. Myatt, W. Rozmus and V. T. Tikhonchuk. *Ion acoustic waves in plasma with collisional electrons*, Phys. Rev. E. **50** 5134-5137 (1994).

V. Yu. Bychenkov, J. Myatt, W. Rozmus and V. T. Tikhonchuk. *Quasihydrodynamic description of ion acoustic waves in a collisional plasma*, Phys. Plasmas **1** 2419-2429 (1994).

Chapter 6 is based on the following work that has yet to be submitted for publication,

J. Myatt, W. Rozmus, V. Yu. Bychenkov, and V. T. Tikhonchuk. *Enhanced fluctuations due to background heat flow.*

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Chapter 1

INTRODUCTION AND OVERVIEW

1.1 Introduction

The transport coefficients for weakly coupled plasmas may be successfully computed from the underlying microscopic kinetic theory [1]. The evaluation of transport coefficients from the microscopic dynamics puts on a firm foundation the phenomenological observations of the laws of Fourier, for heat conduction, Fick for diffusion and Ohm for electrical current. A complete description of the plasma is then afforded by the equations of hydrodynamics, together with the transport coefficients that relate the thermodynamic fluxes to the corresponding thermodynamic forces [1]. For example Fourier's law expresses the flow of heat (the flux) in response to temperature gradients (the corresponding force).

Due in part to the international efforts aimed at Inertial Confinement Fusion (ICF), plasma physics has, and continues to receive much theoretical and experimental attention [2]. ICF plasmas involving laser interaction processes are generally very complicated to model, and usually one would like to keep the hydrodynamic description over the more detailed kinetic theory wherever it is possible, due to to the relative simplicity of the hydrodynamics. The classical hydrodynamic theory requires the mean free path to be short compared with any spatial variation in the hydrodynamic quantities, and the collision frequency must be the highest frequency. The frequent collisions then keep the plasma close to thermodynamic equilibrium. Unfortunately, even when thermal electrons have mean free paths smaller than the

gradient scale, fast electrons responsible for the heat flux may not have, and the need for transport inhibition can arise [3]. The classical transport coefficients predict too large a heat flow, because long mean free path electrons cease to "see" the gradients. This outstanding problem has received much attention, varying from simple clamps on the heat flux, to nonlocal models that introduce frequency and wavenumber dependent transport coefficients either based on numerical solutions to kinetic equations or analytical models. Closure relations that are wavenumber dependent imply that the response of the flux to its corresponding force is delocalized in space.

We propose the use of Thomson scattering (a tool that is presently used in laser plasmas as a temperature diagnostic) for quantitative studies of nonlocal transport processes. Thomson scattering has been used by Zhang *et al.* to investigate classical transport coefficients in CO_2 scattering experiments from strongly collisional Argon plasmas [4, 5]. To date there has been no attempt to quantify the non locality of the transport coefficients by such an experiment. The Thomson scattering cross section is determined by the dynamical form factor for electron density fluctuations, $S(\vec{k},\omega)$. The evaluation of the dynamical form factor requires the solution of the equations of motion for the density fluctuations in the plasma. Linearized hydrodynamics correctly describes the evolution of long wavelength hydrodynamic fluctuations, and hence $S(\vec{k},\omega)$ is then dependent upon the transport coefficients. This was the approach taken by Zhang *et al.* in their investigations [4, 5]. However, an exact result from fluctuation theory is that the phase space fluctuation (the quantity from which all other macroscopic fluctuating quantities can be derived) obeys the exact linearization of the equations for the single particle distribution function [6]. This observation has lead us to formulate a nonlocal hydrodynamic theory for $S(\vec{k},\omega)$, that is based on the solution to the linearized kinetic equation for the fluctuating quantities that is not restricted by the usual validity conditions for hydrodynamics, namely small wavenumber and frequency. The resulting expression for $S(\vec{k},\omega)$ is dependent on

the frequency and wavenumber dependent ion viscosity and electron transport coefficients. We outline regimes of parameters that will enable measurements of electron thermal transport from the Thomson scattering spectrum in the weakly collisional regime of importance to ICF experiments. The same work has lead to the investigation of different background states, in particular fluctuations around a state with a background heat flux.

1.2 Overview of the dissertation

1.2.1 Chapter 2

Chapter 2 starts with a brief examination of scattering processes from neutral fluids, both from liquids and gases, and some of the associated theoretical techniques that have a relevance to our work. The Thomson scattering cross section from plasmas is introduced in Section 2.2, and the important collisionless Vlasov theory for $S(k,\omega)$ in Section 2.2. The predictions of this collisionless theory are the basis of several plasma diagnostics that are examined.

1.2.2 Chapter 3

Chapter 3 outlines the theory of fluctuations in classical plasmas that is obtained from the Bogoliubov-Born-Kirkwood-Green-Yvon (BBGKY) hierarchy by low order expansions in the plasma parameter. The original work due to Rostoker [7] is described, together with more recent simplifications [6]. Of these simplifications it is the conclusion that the phase space correlation function (that is the basic quantity for the calculation of all correlations) obeys the exact linearization of the kinetic equation for the one particle distribution function that opens the way for the application of

the techniques of kinetic theory to the theory of fluctuations. This is the motivation for our approach to calculating the dynamic form factor in Chapter 5.

1.2.3 Chapter 4

Chapter 4 demonstrates the reduction of the kinetic equation for the one particle distribution function to hydrodynamics that is the basis of classical transport theory. The Hilbert and Chapman-Enskog methods that achieve this are described for a neutral gas, together with the necessary validity conditions in Section 4.1. The application of the Chapman-Enskog method to plasmas is presented in some detail in Section 4.2 together with the values for the classical transport coefficients in Section 4.2.1.

The classical transport coefficients are often inadequate in describing heat flow in many ICF relevant plasmas. Some nonlocal transport models that have been proposed are discussed in Section 4.3. We describe the nonlocal electron transport model due to Bychenkov *et al.* [8] in Section 4.4 and a model for ion transport based on a frequency dependent closure in Section 4.5.

1.2.4 Chapter 5

Chapter 5 contains the derivation of the dynamical form factor $S(\vec{k},\omega)$ from the nonlocal transport models described in Sections 4.4 and 4.5 of Chapter 4. The application of the nonlocal theory in the limit of collisional ions is given in Section 5.3.1 and in the limit of collisionless electrons in Section 5.3.2.

1.2.5 Chapter 6

Chapter 6 is an extension of the theory for the dynamical form factor developed in Chapter 5 from the nonlocal transport models of Sections 4.4 and 4.5, in order to include plasmas that carry a heat flux. A derivation of the heat carrying background state that satisfies the electron kinetic equation is given in Section 6.2, and an analysis of the results for $S(k,\omega)$ is given in Section 6.4.

Chapter 2

SCATTERING EXPERIMENTS

Several different scattering processes from plasmas and neutral fluids are dependent on a rather simple property of the medium: the space and time Fourier transform of the time-dependent density correlation function, known as the dynamic form factor. The different time and length scales of the density fluctuations in neutral liquids and gases probed by neutron and laser light scattering are examined together with some of the relevant theoretical work that has been developed from both statistical mechanics and thermodynamics. With the appropriate theory, scattering experiments from neutral fluids are seen to be a very useful probe of the properties of the medium.

Thomson scattering of an electromagnetic wave from plasmas has been a very important tool since the advent of plasma physics research. Plasmas are in many ways quite different from neutral fluids, and much of the work concerned with collisionless plasmas bears little resemblence to the fluid theories. However, one is still concerned with the evaluation of the same theoretical quantity, and for collisional plasmas we see that some of the ideas from neutral fluids have parallels, and that ideas may be transferred from one field to the other.

2.1 Scattering from neutral fluids

2.1.1 Slow neutron scattering

In 1954 Van Hove demonstrated that the measurement of coherent inelastic scattering of neutrons from a liquid can be used to probe the dynamics of fluctuations in the fluid, namely the dynamic form factor, $S(\vec{k},\omega)$ [9]. The probability P for a neutron to transfer the momentum $\hbar\vec{k}$, and energy $\hbar\omega$ to the fluid is given by

$$P(\vec{k},\omega) = N|\,V(k)\,|^2 S(\vec{k},\omega), \tag{2.1}$$

where N is the number of nucleons in the scattering volume, and $V(k)$ is the potential of interaction between the nucleus and the probe neutron. The dynamic form factor is the time and space Fourier transform of the Van Hove function $G(\vec{x},t)$,

$$S(\vec{k},\omega) = \int d\vec{x}dt\, \exp i(\vec{k}\cdot\vec{x} - \omega t) G(\vec{x},t). \tag{2.2}$$

The Van Hove function G is the time-dependent generalization of the pair distribution function

$$G(\vec{x},t) = \frac{1}{\langle n\rangle}\langle n(0,0)n(\vec{x},t)\rangle, \tag{2.3}$$

where $n(\vec{x},t)$ is the microscopic particle density

$$n(\vec{x},t) = \sum_{i=1}^{N} \delta[\vec{x} - \vec{x}_i(t)], \tag{2.4}$$

and $\vec{x}_i(t)$ is the trajectory of the i^{th} particle. The average in (2.3) can be regarded as an ensemble average in the usual sense [10]. Cold neutrons from a reactor probe liquids with characteristic momentum transfers and energies that corresponds to wavevectors and frequencies in liquids that are much higher than those for which hydrodynamics is valid, $k \sim 10^{12}$ cm^{-1} and $\omega \sim 10^{13}$ sec^{-1}, compared to collision frequencies in liquids that are of the order of 10^9 to 10^{10} sec^{-1}. These wavenumbers and frequencies are associated with the kinetic regime. Neutron scattering is one of many dynamic processes that can be discussed in terms of $S(\vec{k},\omega)$.

2.1.2 Rayleigh scattering

Komarov and Fisher have adapted Van Hove's neutron scattering results to light scattering [11]. For Rayleigh scattering of light from density fluctuations in a liquid the scattered intensity is given by

$$I(\vec{R},\omega) = I_0 \frac{\beta^2 k_0^4 N}{2\pi R^2} \sin^2\phi\, S(\vec{k},\omega), \tag{2.5}$$

where I is the intensity of observed light at the position $\vec{R}$, having been scattered from the origin. The shift in frequency of the scattered light is given by ω and $\vec{k}$ is the change in the wavenumber of the scattered light $\vec{k}_s$, from that of the incident light in the medium $\vec{k}_0$, so that $\vec{k} = \vec{k}_s - \vec{k}_0$. I_0 is the incident intensity of the probe beam that is monochromatic and plane polarized with the angle ϕ between the position vector $\vec{R}$ and the electric vector of the incident wave. N is the number of particles in the scattering volume of polarizability β. The dynamic form factor $S(k,\omega)$ is given in terms of the Van Hove function by Eqs. (2.2, 2.3) The wavenumber $\vec{k}$ of the fluctuations probed depends of the scattering angle θ according to the simple formula

$$k = 2k_0 \sin\left(\frac{\theta}{2}\right). \tag{2.6}$$

Rayleigh scattering from liquids

Taking a He–Ne laser as a probe, having a wavelength of 0.6328 μm and considering a scattering angle $\theta = 60°$, the probed wavenumber k calculated with Eq. (2.6) is of the order $k \sim 10^5$ cm^{-1}. In a liquid this length scale is much larger than a mean free path and the sound frequencies are small compared to collision frequencies of 10^9-10^{10} sec^{-1}. These orderings correspond to the hydrodynamic regime, in which the liquid may be adequately described by a set of partial differential equations for the conserved macroscopic variables: number density $n(\vec{x},t)$, hydrodynamic velocity

$\vec{u}(\vec{x},t)$, and temperature $T(\vec{x},t)$. For a neutral fluid these are the Navier–Stokes equations.

A method of describing fluctuations by macroscopic equations, solved as an initial value problem with initial correlations appropriate to thermal equilibrium, has been demonstrated by Onsager for systems in thermal equilibrium [12]. For a fluid, fluctuations of the macroscopic variables δn, $\delta\vec{u}$ and δT that are spontaneously excited at some initial time, will decay according to the same equations as do ordinary induced disturbances, i.e. in a neutral fluid these are the Navier-Stokes equations. This "Onsager regression" was originally derived by thermodynamic arguments [12]. The application to fluids was suggested by Landau and Placzek [13], and they were the first to predict the widths of the lines in the scattered spectrum are determined by the lifetimes of the density fluctuations δn obtained from linearized hydrodynamics [13].

Mountain has applied this theory [14] and summarized some of the results, that we repeat here using Mountain's notation. The Onsager method results in the solution for the dynamic form factor $S(\vec{k},\omega)$ in terms of the static structure factor $S(k)$, which takes the form [14],

$$S(k,\omega) = S(k)\sigma(k,\omega), \tag{2.7}$$

where

$$\sigma(k,\omega) = (1-\gamma^{-1})\frac{2\lambda k^2/\rho_0 c_p}{(\lambda k^2/\rho_0 c_p)^2+\omega^2} + \gamma^{-1}\left[\frac{\Gamma k^2}{(\Gamma k^2)^2+(\omega+C_0 k)^2}+\frac{\Gamma k^2}{(\Gamma k^2)^2+(\omega-C_0 k)^2}\right]. \tag{2.8}$$

In the above, Γ is the acoustic wave attenuation

$$\Gamma = \frac{1}{2}\left[\frac{\eta_B+4/3\eta_S}{\rho_0}+\frac{1}{\rho_0}\left(\frac{\lambda}{c_v}-\frac{\lambda}{c_p}\right)\right], \tag{2.9}$$

which is dependent upon the bulk and shear viscosities η_B and η_S, and the thermal conductivity λ. Also, ρ_0 is the density, γ is the ratio of specific heats, $\gamma = c_p/c_v$ and C_0 is the sound speed.

The structure of the spectrum of scattered light predicted by (2.7, 2.8) consists of three peaks. A central diffusive Rayleigh peak at $\omega = 0$, and a pair of peaks known as the Mandelshtam-Brillouin doublet at $\omega = \pm C_0 k$. The Rayleigh peak corresponds to a diffusive mode that is non-propagating and is therefore unaffected by viscous effects, η_B and η_S. The Mandelshtam-Brillouin doublet is the scattering of light from thermal sound waves, as first predicted by Brillouin in 1922 [15], with the two peaks corresponding to waves propagating in opposite directions. An analysis of the scattered spectrum allows thermodynamic properties of the fluid to be determined, as the ratio of the light scattered in the central peak, I_1 to the Brillouin lines, I_2 is equal to

$$\frac{I_1}{2I_2} = \gamma - 1, \tag{2.10}$$

as can be verified by integrating the expression for $S(k,\omega)$ over all ω. A measurement of the separation of the Brillouin doublet also gives the speed of the thermal sound wave, $C_0 = \sqrt{\gamma(\partial p/\partial \rho)_T}$. The widths of the Mandelshtam-Brillouin lines are a measure of the ion acoustic attenuation Γ (2.9), which yields information on the transport properties of the liquid, i.e. the thermal conduction, λ and the bulk and shear viscosities, η_B and η_S. A measurement of the width of the Rayleigh peak will also give the thermal conduction.

The result that transport properties may be inferred from the dynamic form factor $S(\vec{k},\omega)$ is not surprising, as there is a well known connection between fluctuations in equilibrium and the dissipative properties of the medium in terms of the "fluctuation-dissipation theorem" [16]. The result is however very useful for measurements of transport properties that are difficult by more conventional means, such as measurements of thermal diffusivity around the critical point of CO_2 [14].

Rayleigh scattering from gases

If one now considers Rayleigh scattering from neutral gases rather than liquids, then the probed wavenumbers and frequencies become comparable to the mean free path and collision time. The hydrodynamic description would then be expected to fail. In this case the time dependence of the Van Hove correlation function cannot be determined from hydrodynamics, and must be determined from the kinetic theory. Using complicated diagrammatic methods Van Leeuwen and Yip were able to show that for classical gases Van Hove's correlation function (2.3) may be calculated via the linearized Boltzmann equation [17].

$$G(\vec{x},t) = \int d\vec{v}_0 d\vec{v}\, \Gamma(\vec{x}_0,\vec{v}_0,t_0;\vec{x},\vec{v},t), \tag{2.11}$$

where $\Gamma(\vec{x}_0,\vec{v}_0,t_0;\vec{x},\vec{v},t)$ satisfies the linearized Boltzmann equation in the $\vec{x}$, $\vec{v}$, t variables,

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{x}} + K\right)\Gamma = 0, \tag{2.12}$$

and the linear Boltzmann collision operator has been denoted by $-K$. The initial condition is given by,

$$\Gamma(\vec{x}_0,\vec{v}_0,t_0;\vec{x},\vec{v},t_0) = \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(\frac{mv^2}{2T}\right)\delta(\vec{x}-\vec{x}_0)\delta(\vec{v}-\vec{v}_0). \tag{2.13}$$

This effectively becomes a kinetic version of Onsager regression, as explained in some detail by Hinton [18]. Hinton, as well as giving a clearer derivation of Van Leeuwen's results [17] from a BBGKY (N. N. Bogoliubov, M. Born, G. Kirkwood, H.S. Green and J. Yvon) approach, was also able to reduce this description in terms of the function Γ to the linearized Navier-Stokes equations for the fluctuating hydrodynamic variables n, $\vec{u}$, and T, by a modification of the Chapman-Enskog method [18]. Hinton's results have put on a firm theoretical foundation the work of Mountain [14] and have also extended the validity of the Onsager method from that of absolute thermal equilibrium to small deviations from a local equilibrium state.

Using the theoretical justification of Van Leeuwen, Ranganathan and Yip solved the linearized Boltzmann equation (2.12) as an initial value problem [19]. They have obtained an exact analytic solution for $S(\vec{k},\omega)$ by assuming Maxwell molecules, the form of which permit an exact evaluation of the spectrum of the linear Boltzmann collision operator [10]. Using these solutions Nelkin and Yip have considered Rayleigh scattering from fluctuations in an Argon gas as a test of the linearized Boltzmann equation. They compared the solutions for $S(\vec{k},\omega)$ from Ref. [19] with $S(\vec{k},\omega)$ calculated from the linearized Navier-Stokes equations of hydrodynamics [14], and gave an estimate of the expected results. Such an experiment was actually performed by Greytak and Benedek [20], who were able to modify the wavenumber of the probed fluctuations from the hydrodynamic regime into the kinetic regime by changing the scattering angle from the forwards 10° to the near backwards direction 170°. Unfortunately their success was only limited due to the insufficient resolution of their apparatus.

2.2 Thomson scattering from plasmas

If a neutral gas is heated sufficiently to cause it to ionize, forming a plasma, another scattering process is possible. An incident electromagnetic wave on a plasma will cause the electrons to accelerate in the electric field. The accelerating charges will then emit radiation in all directions except in that of the incident wave. If the photon energy of the incident electromagnetic wave $\hbar\omega$ is much less than the rest energy of the electron mc^2, then the process is known as Thomson scattering.

The time-averaged Thomson scattered power P_s per unit solid angle Ω per unit frequency $\omega/2\pi$ is given by [21],

$$\frac{dP_s}{dt d\Omega d\omega/2\pi} = \frac{P_i r_0^2}{A}\left(1 - \sin^2\theta\cos^2\mid \phi - \phi_0 \mid\right) N_e S(\vec{k},\omega), \qquad (2.14)$$

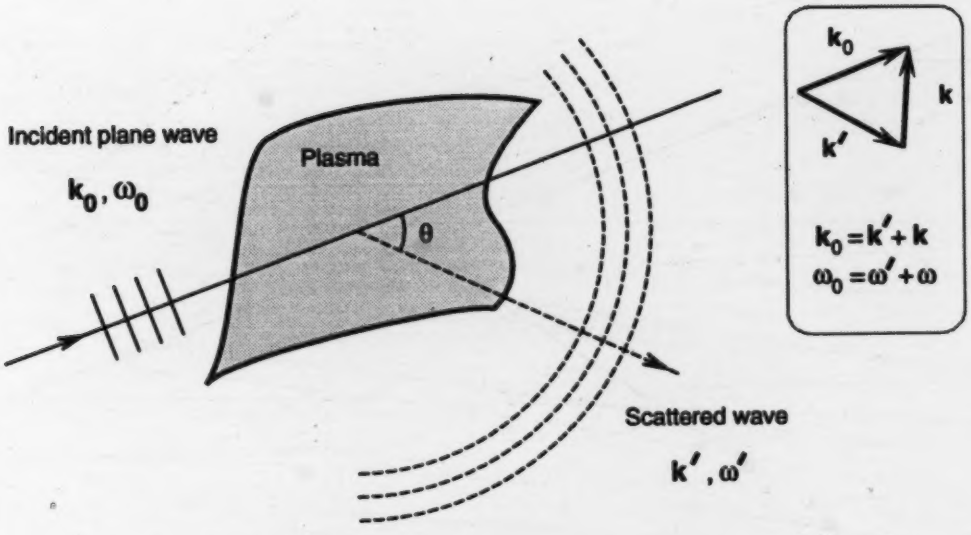

Figure 2.2.1: Schematic of a Thomson scattering experiment

where $r_0 = e^2/mc^2$ is the classical electron radius, P_i is the incident power, A is the cross sectional area of the incident beam, N_e is the number of electrons in the scattering volume, V, $\vec{k} = \vec{k}_0 - \vec{k}'$ and $\omega' = \omega_0 - \omega$ are the momentum and energy transfer, i.e. the difference in the wavevector and frequency between the probe $(\vec{k}_0, \omega_0)$ and scattered $(\vec{k}', \omega')$ electromagnetic waves. θ is the scattering angle, and $\phi - \phi_0$ is the angle between the incident electric field vector, and $\vec{k}'$ in a plane perpendicular to $\vec{k}_0$, see Fig. 2.2.1. The dynamic form factor (2.2, 2.3) averaged over time T and the scattering volume

$$S(\vec{k}, \omega) = \lim_{V \to \infty, T \to \infty} \frac{1}{VT} \frac{\langle n_e(\vec{k}, \omega) n_e^*(\vec{k}, \omega) \rangle}{\langle n_e \rangle}, \tag{2.15}$$

is for fluctuations in electron density

$$n_e(\vec{k}, \omega) = \int d\vec{x} dt \, \exp i(\vec{k} \cdot \vec{x} - \omega t) \sum_{i=1}^{N_e} \delta[\vec{x} - \vec{x}_i(t)], \tag{2.16}$$

as the contribution to the scattering from the ions is negligible due to their larger mass and hence smaller acceleration in the laser field. If the incident light is unpolarized,

the replacement $\cos^2(\phi - \phi_0) \rightarrow 1/2$ can be made in (2.15). A derivation of the Thomson scattering cross section is provided in Appendix A.

It was the work on the scattering of radio waves from the ionosphere that stimulated the development of Thomson scattering as a more general diagnostic for plasmas. In particular, it was Gordon who first suggested that ionospheric electron plasma density and temperature could be measured by "incoherent" radar backscatter [22]. Incoherent in this context refers to the parameter $\alpha = 1/k\lambda_D < 1$, which means the scattering electrons display single particle behaviour [21]. When the plasma is probed on a scale larger than a Debye length $\alpha > 1$, the plasma can show collective behaviour. The dynamic form factor for independent electrons, relevant to incoherent scattering is

$$S(\vec{k}, \omega) = 2\pi \int d\vec{v} f_e(\vec{v}) \delta(\omega - \vec{k} \cdot \vec{v}), \tag{2.17}$$

where $f_e(\vec{v})$ is the electron distribution function normalized to unity. It is apparent from (2.17) that a Thomson scattering measurement from a Maxwellian plasma would therefore reflect the electron thermal spread, and hence allow a determination of electron temperature.

With the motivation of explaining ionospheric experiments the general form of the scattered spectrum of a plasma was determined independently in 1960 by Fejer [23], Renau [24], Dougherty and Farley [25], and Salpeter [26]. In these theories, the fluctuations caused by particle discreteness are described by the linearized Vlasov equation. The Vlasov equation adequately describes the collective effects of the plasma, but ignores particle collisions that are responsible for the approach to equilibrium. This is then the completely opposite extreme to the hydrodynamic expression (2.7, 2.8) for neutral fluids, where collisions are dominant. In this respect plasmas are fundamentally different from neutral fluids in that they can support collective behaviour even in the absence of particle collisions.

Dynamic form factor for a collisionless plasma

The collisionless expression for the dynamic form factor $S(\vec{k},\omega)$ in an electron-ion plasma is

$$S(\vec{k},\omega) = \frac{2\pi}{k}\left|1-\frac{\chi_e}{\epsilon}\right|^2 \bar{f}_e\left(\frac{\omega}{k}\right) + \frac{2\pi Z}{k}\left|\frac{\chi_e}{\epsilon}\right|^2 \bar{f}_i\left(\frac{\omega}{k}\right), \tag{2.18}$$

where $\bar{f}(\omega/k)$ are the one-dimensional distribution functions in the direction of $\vec{k}$ evaluated at the phase velocity $v_{ph} = \omega/k$, and normalized to unity. Z is the ionization, $\epsilon(k,\omega) = 1+\chi_e+\chi_i$ is the dielectric function and χ_e and χ_i are the electron and ion partial susceptibilities that are defined and evaluated for a Maxwellian plasma in Appendix B.1. The expression for $S(\vec{k},\omega)$ is evaluated for a Maxwellian plasma in Appendix B.2, in which the electrons and ions are allowed to have separate temperatures and mean velocities. Equation (2.18) for $S(\vec{k},\omega)$ consists of two terms. The first term in Eq. (2.18) describes the scattering from both free electrons (the incoherent part) and from electrons that are in the screening cloud of other electrons (electrons behaving coherently). The second term describes the scattering from electrons that are in the screening cloud of the ions. For an equal temperature plasma, where the ion acoustic mode is heavily Landau damped, the ion feature shows the ion thermal spread, even though it is the electrons that do the scattering, see Fig. 2.2.2. This is consistent with the results of the the experiment of Bowles which gave the first direct confirmation of collective behaviour in a plasma [27].

Peaks appear in the spectrum $S(\vec{k},\omega)$ (2.18) whenever Re $\epsilon = 0$, corresponding to the natural modes of the plasma. For a non-magnetized plasma these correspond to ion acoustic and Langmuir waves. In addition to the low frequency spectrum that we have briefly discussed, scattering from fluctuations of much higher frequency reveals the Langmuir resonance, see Figure 2.2.3. Measurement of the electron plasma frequency,

$$\omega_{pe}^2 = \frac{4\pi n_e e^2}{m_e} \tag{2.19}$$

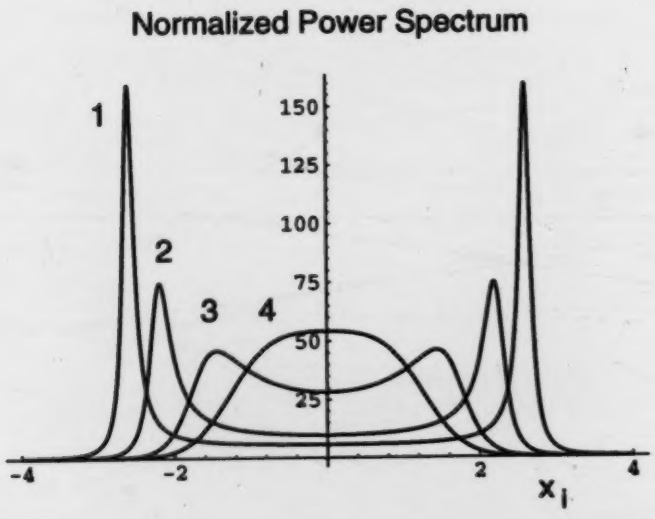

Figure 2.2.2: The dependence of the normalized power spectrum $S(k,\omega)$ as a function of x_i, where $x_i = v_{ph}/v_{Ti}$.The curves 1,2,3, and 4 correspond to the temperature ratios of $ZT_e/T_i = 10, 6, 2, 0.5$ respectively. Notice the development of the ion acoustic resonance and its dependence on ZT_e/T_i. For ion acoustic waves $c_s \approx (ZT_e/T_i)^{1/2} v_{Ti}$.

also allows for the determination of the electron density. The first measurement of the electron density profile in the ionosphere was achieved by Perkins [28] using this method at the Arecibo radar facility.

2.2.1 Thomson scattering and laser plasmas

With the advances in laser technology, Thomson scattering has become a standard diagnostic for electron temperature and density in many laboratory devices, especially in magnetic confinement experiments, after modifications to include the ambient magnetic field. In laser plasmas, Thomson scattering has become a standard technique for studying enhanced fluctuations (cf. e.g. Refs. [29, 30]), and thermal fluctuations [31, 32, 33, 34, 35, 36]. The applications to laser plasmas, for example in the research of x-ray lasers or ICF are made difficult due to the small scattering volume,

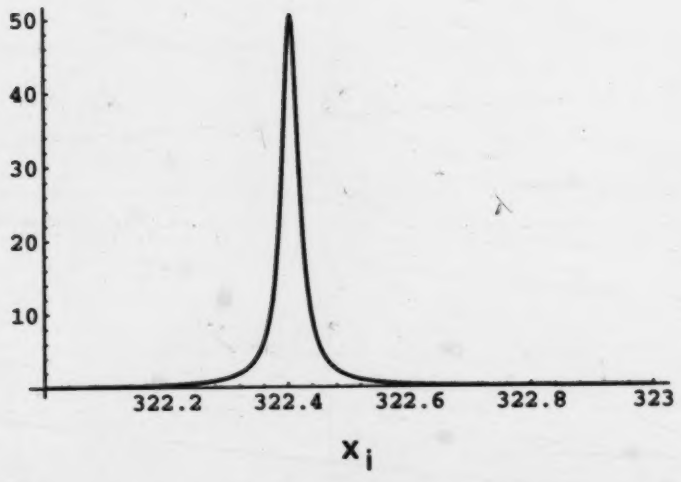

Figure 2.2.3: Looking at $S(k,\omega)$ for larger values of x_i we find one more resonance. Making use of the relations $x_e = (ZT_e/T_i)^{-1/2}(Zm_e/Am_p)^{1/2}x_i$ and $x_e \approx 1/k\lambda_{De}$ at $\omega = \omega_{pe}$ we identify this mode to be that of *electron plasma waves*. The temperature ratio ZT_e/T_i in this plot is 2 and $k\lambda_{De} = 1/5$.

low intensity of the scattered light and fast time evolution of the plasma, often on the picosecond time scale. In spite of the complexity of laser plasma interactions, the simple collisionless theory (2.18) has been used as the theoretical input for the characterization of near equilibrium x-ray laser and ICF type plasmas. The studies [33, 34, 35, 36] have centered around the low frequency part of the scattered spectrum shown in Fig. 2.2.2.

The peaks that develop in the low frequency scattered spectrum for increasing ZT_e/T_i correspond to the scattering of the light from the thermal ion acoustic fluctuations. These ion acoustic fluctuations are heavily Landau damped unless ZT_e/T_i is appreciable, as the phase velocity of the wave falls in the bulk of the ion distribution, where Landau damping is large. A measurement of the peak separation, which is equal to twice the ion acoustic frequency ω_s can be used to infer electron temperature

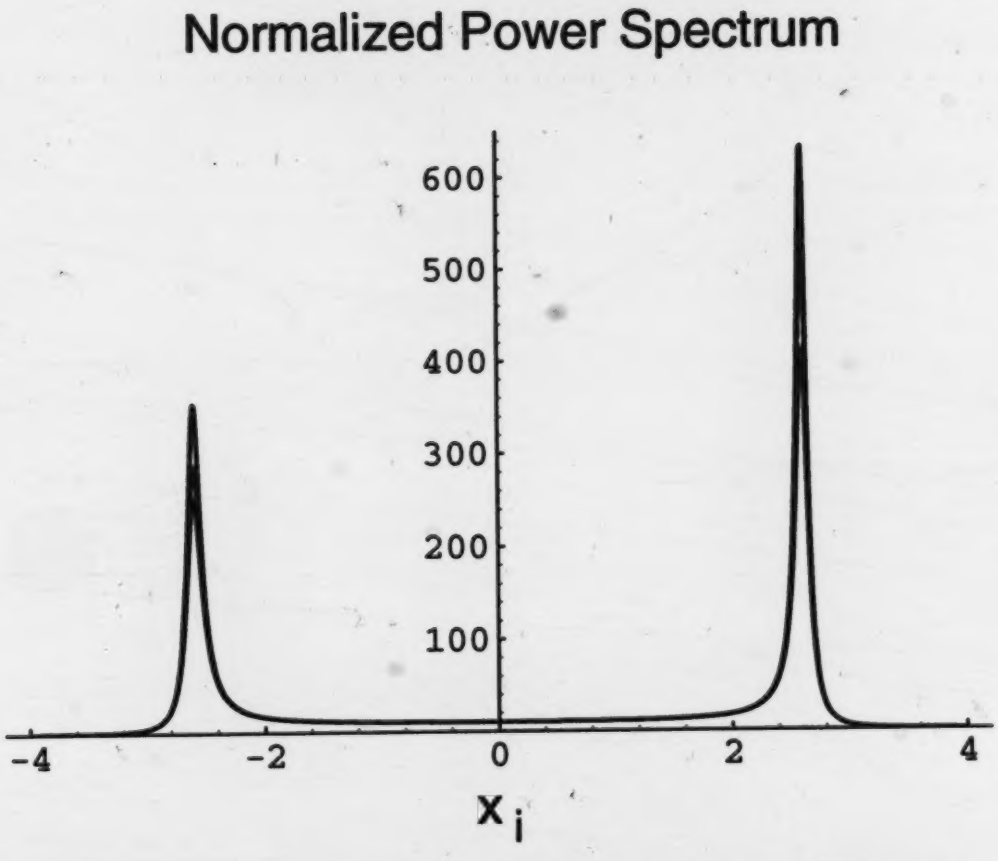

Figure 2.2.4: Here we see three cases of increasing electron drift $\vec{u}_e$ corresponding to $u_e = (0.2, 0.8, 1.2)\, c_s$. The height of the peaks is asymmetric due to the unequal electron Landau damping of the two waves traveling in opposite directions to the drift . The wave traveling in the direction of the drift becomes unstable for drifts of the order of the sound speed, c_s.

If the ionization Z is known,

$$\omega_s = kv_s, \quad \text{where} \quad v_s \approx \sqrt{\frac{ZT_e}{m_i}}\left(1+3\frac{T_i}{ZT_e}\right)^{1/2} \tag{2.20}$$

for a collisionless plasma. For moderate values of ZT_e/T_i, the width of the ion acoustic peaks are predominantly determined by ion Landau damping

$$\frac{\gamma_i}{\omega_s} \approx 1.1\left(\frac{T_i}{ZT_e}\right)^{7/4}\exp-\left(\frac{T_i}{ZT_e}\right)^2, \tag{2.21}$$

where γ_i is the ion Landau damping rate. Given that the ion acoustic frequency ω_s is already known, this measurement can be used to obtain an estimate for T_i [36]. Ion hydrodynamic flow velocity $\vec{u}_i$ in the direction of the probed $\vec{k}$ vector can be measured from the Doppler shift in the whole spectrum. The electron drift velocity $\vec{u}_e$ relative to the ion flow can also be measured as the peaks become asymmetric in height due to the unequal electron Landau damping of the counter propagating ion acoustic fluctuations, see Fig. 2.2.4.

2.2.2 Collisional plasmas

The collisionless assumption of the Vlasov theory can break down for some ICF and x-ray laser plasmas. High ionization Z, and density n, act to increase the frequency of collisions, making it necessary to include collisional damping of the fluctuations for a detailed agreement of Thomson scattering line shapes [37]. From a theoretical point of view, the introduction of collisions into the theory of fluctuations can allow for a hydrodynamic description for sufficiently long wavelengths, of the kind we have described for neutral fluids in Section 2.1.2. Although the hydrodynamic limit is not relevant to the experiments of Refs. [31, 32, 33, 34, 35, 36], Zhang *et al.* have conducted experiments on more strongly collisional plasmas [4, 5]. Zhang *et al.* have found the Thomson scattered spectrum of a CO_2 laser from a strongly collisional Argon arc plasma ($n_e = 10^{17}$ cm$^-3$, $T_e \sim 2$ eV) can be well described by the dynamic

form factor, calculated from the linearized fluid equations of Braginskii [1]. As a result, they have been able to deduce the values of the transport coefficients in the same way as we have described in Section 2.1.2 for the Landau-Placzek formula (2.7, 2.8). This work was also the first to observe the entropy peak, that is the counterpart of the Rayleigh peak in neutral fluids and does not exist in a collisionless plasma described by Eq. (2.18).

2.2.3 Extension of the theory to include collisions

Rostoker has laid out a systematic BBGKY approach for calculating fluctuations in plasmas as a perturbation in the plasma parameter [7]. To the lowest order this theory leads to the collisionless expression (2.18), as demonstrated by Rosenbluth and Rostoker [38]. In order to describe the effects of collisions it is necessary to go to the next order in the expansion. In the next chapter we will see how more recent developments have simplified this theory, and lead to a linear kinetic for the fluctuation of the phase space density, in the same way as Van Leeuwen [17] and Hinton [18] have found in neutral gases.

Chapter 3

FLUCTUATIONS IN CLASSICAL PLASMAS

3.1 Formalism of two-time correlations

We introduce the 6N dimensional Γ-space given by $\{X_i\} \equiv (X_1, X_2, \ldots, X_N)$, where $X_i = (\vec{x}_i, \vec{v}_i)$, $(i = 1, \ldots, N)$ so that the whole N-body classical system is a single point in this space called a *system point* [10]. The point then wanders through this space with time as determined by the dynamical equations. In order to calculate the macroscopic expectations of microscopic functions of the phase space, such as the number density $\langle n(\vec{x}, t)\rangle$ or density correlation functions $\langle n(\vec{x}, t)n(\vec{x}', t')\rangle$ for example, we need to assign a weight that the system is in the volume $d\{X_i\} = dX_1 \ldots dX_N$ centered around a particular microstate $\{X_i\}$ in the Γ-space at a given time t. This is achieved in the usual way through the construction of an ensemble of N replicas of the system (which can be taken to infinity). The symbol N is also used to denote the total number of particles in the sysetm, although it should be obvious from the context which is intended. The density of system points of the ensemble in the Γ-space

$$D_1(\{X_i\}, t) = \frac{dN}{d\{X_i\}} \tag{3.1}$$

times the small volume element $d\{X_i\}$: $D_1(\{X_i\}, t)d\{X_i\}$, can be identified with the probability that the system is in the volume $d\{X_i\}$ about the phase point $\{X_i\}$ at the time t, after a suitable normalization

$$\int d\{X_i\}\, D_1(\{X_i\}, t) = 1. \tag{3.2}$$

For calculating two-time expectations we will follow Rostoker [7] and introduce the two-time generalization of D_1, $D_2(\{X_{0i}\}, t_0; \{X_i\}, t) d\{X_{0i}\}\, d\{X_i\}$ which is the probability that the system is in the volume $d\{X_{0i}\}$ about the phase point $\{X_{0i}\}$ at the time t_0 and then also in the volume $d\{X_i\}$ about the phase point $\{X_i\}$ at the later time t. With $D_1(\{X_i\}, t)$ we are able to calculate the expectation of one-time phase functions $A(X; \{X_i\})$,

$$\langle A(X,t)\rangle = \int d\{X_i\} D_1(\{X_i\}, t) A(X; \{X_i\}), \tag{3.3}$$

and two-time phase functions with $D_2(\{X_i\}, t; \{X_i'\}, t')$,

$$\langle A(X,t)B(X',t')\rangle = \int d\{X_i\} d\{X_i'\} D_2(\{X_i\}, t; \{X_i'\}, t') A(X; \{X_i\}) B(X'; \{X_i'\}). \tag{3.4}$$

In this way we are able to calculate expectations and correlations for macroscopic observables, provided that some means exists for calculating the joint probabilities D_1 and D_2 for the system.

3.1.1 Liouville equation for the Joint Probability Density

The evolution of a trajectory of a system point in Γ-space is unique for a given initial condition $\{X_i\} = \{X_{i0}\}$ at $t = 0$, and hence the trajectories in the Γ-space cannot cross. Considering a volume element of phase space $d\{X_i\}$, we come to the conclusion that system points interior to this volume must remain interior for all times. Taken together with the fact that time evolution is a canonical transformation, and hence the volume occupied by these points is also unchanged in time

$$\begin{aligned} t &\mapsto t' \\ dN &\mapsto dN', \\ d\{X_i\} &\mapsto d\{X_i'\}. \end{aligned}$$

we conclude that the points flow like an incompressible fluid in the phase space. On a system trajectory the density of points in the phase space is constant

$$\frac{dD}{dt} = 0. \tag{3.5}$$

The above is a statement of Liouville's theorem [10]. Therefore, by (3.5) D_1 and D_2 satisfy a grand continuity equation, the "Liouville equation",

$$\left(\frac{\partial}{\partial t} + \sum_{i=1}^{N} \vec{v}_i \cdot \frac{\partial}{\partial \vec{x}_i} + \vec{F}_{ex}(X_i,t) \cdot \frac{\partial}{\partial \vec{v}_i} \right.$$
$$\left. - \frac{e_i}{m_i} \sum_{j \neq i}^{N} \frac{\partial}{\partial \vec{x}_i} \frac{e_j}{|\vec{x}_i - \vec{x}_j|} \cdot \frac{\partial}{\partial \vec{v}_i} \right) \left\{ \begin{array}{c} D_1(\{X_i\},t) \\ D_2(\{X_i\},t;\{X'_i\},t') \end{array} \right\} = 0, \tag{3.6}$$

where $\vec{F}_{ex}$ is the external force acting on the system, and the interactions between the particles are given by Coulomb's law. The initial condition for D_2 is given by

$$D_2(\{X_i\},t;\{X'_i\},t) = D_1(\{X_i\},t)\delta(\{X_i\} - \{X'_i\}). \tag{3.7}$$

Knowledge of the most general solution to this equation is equivalent to a knowledge of all the orbits of the system, which is clearly an impossible task. The Liouville equation (3.6) is however our starting place for deriving approximate solutions for the joint probability densities D_1 and D_2, and hence a theory of fluctuations.

3.1.2 Kinetic theory of fluctuations

Much of the information contained in D_1 and D_2 is redundant. We shall see that for the calculation of most practical observables (3.3, 3.4) we only need to evaluate certain moments of D_1 and D_2. For two-time fluctuations the basic theoretical quantity is $\langle \delta f^\alpha(X,t) \delta f^{\alpha'}(X',t') \rangle$, where δf^α is the deviation of the Klimontovich microdensity from its ensemble average [39]

$$\delta f^\alpha = \langle f^\alpha \rangle - f^\alpha \quad \text{where} \quad f^\alpha(X;\{X_i\}) = \frac{1}{n_\alpha} \sum_{i}^{N_\alpha} \delta[X - X_i(t)]. \tag{3.8}$$

The subscript (or superscript) α is a species label that we will assume to be either electrons or ions $\alpha = e, i$. The ensemble average in (3.8) is over the Liouvilliain distribution, D_1

$$\langle f^\alpha(X,t)\rangle = \int d\{X_i\}\, D_1(\{X_i\},t) f^\alpha(X;\{X_i\}), \tag{3.9}$$

where D_1 satisfies the Liouville equation (3.6) in the X_i variables. Making use of the definitions (3.4, 3.8) the correlation of the phase space density $f^\alpha(X;\{X_i\})$ (3.8) is given by

$$\begin{aligned}
&\langle f^{\alpha_0}(X_0,t_0) f^\alpha(X,t)\rangle = \\
&\quad \int d\{X_i\} d\{X_i'\} D_2(\{X_i\},t;\{X_i'\},t_0) f^\alpha(X,t;\{X_i\}) f^{\alpha_0}(X_0,t_0;\{X_i'\}) \\
&\quad = \frac{V^2}{N} \int d\{X_i\} d\{X_i'\} D_2(\{X_i\},t;\{X_i'\},t_0) \delta[X_0 - X_1'(t_0)] \delta[X - X_1(t)] \\
&\quad + V^2 \int d\{X_i\} d\{X_i'\} D_2(\{X_i\},t;\{X_i'\},t_0) \delta[X_0 - X_1'(t_0)] \delta[X - X_2(t)],
\end{aligned} \tag{3.10}$$

where there are two terms in (3.10) corresponding to pairings of the same, and different particles in the double sum (3.8), i.e particle 2 is different from particle 1. We will write these two terms as

$$\langle f^{\alpha_0}(X_0,t_0) f^\alpha(X,t)\rangle = \delta_{\alpha_0\alpha} \Omega_1^{\alpha_0\alpha}(X_0,t_0;X,t) + F_1^{\alpha_0\alpha}(X_0,t_0;X,t). \tag{3.11}$$

The correlation of the phase space fluctuation δf^α (3.8) can be written in terms of the above equation (3.11)

$$\langle \delta f^\alpha \delta f^{\alpha'}\rangle = \langle f^\alpha f^{\alpha'}\rangle - f_1^\alpha f_1^{\alpha'} \tag{3.12}$$

since we have by definition $\langle \delta f^\alpha\rangle = 0$, and we have introduced $f_1^\alpha = \langle f^\alpha\rangle$ as the ensemble average (3.3) of f^α (3.8). We give (3.12) the name $\Gamma_1^{\alpha\alpha'}$ which is the combination of the moments of D_1 and D_2.

$$\begin{aligned}
\Gamma_1^{\alpha\alpha'}(X,t;X',t') = \delta_{\alpha\alpha'} \Omega_1^{\alpha\alpha'}(X,t;X',t') + F_1^{\alpha\alpha'}(X,t;X',t') \\
- f_1^\alpha(X,t) f_1^{\alpha'}(X',t')
\end{aligned} \tag{3.13}$$

The moments $\Omega_1^{\alpha\alpha'}$ and $F_1^{\alpha\alpha'}$ implied in (3.10) were first introduced by Rostoker and will be referred to as the Rostoker functions [7]. The Rostoker functions are given by

$$\Omega_1^{\alpha_1\alpha_1}(X_1,t;X'_1,t') = \frac{V^2}{N_{\alpha_1}}\int D_2(\{X_i\},t;\{X'_i\},t'),dX_2\cdots dX_N dX_2'\cdots dX_N' \tag{3.14}$$

$$F_1^{\alpha_1\alpha'_2}(X_1,t;X'_2,t') = V^2\int D_2(\{X_i\},t;\{X'_i\},t')dX_2\cdots dX_N dX_1' dX_3'\cdots dX_N' \tag{3.15}$$

From Eq. (3.14) we can see that Ω_1 has the interpretation of a test particle function. $\Omega_1(X_1,t;X_1',t')$ gives the probability of a given particle 1 being at $(\vec{x},\vec{v})$ at time t and then that *same* particle being at $(\vec{x}',\vec{v}')$ at the later time t'. $F_1(X_1,t;X'_2,t')$ has the interpretation of a field particle function. This gives the probability of a given particle 1 being found at $(\vec{x},\vec{v})$ at time t and then a *different* particle 2 at $(\vec{x}',\vec{v}')$ at the later time t'. The uncorrelated part of F_1 has also been subtracted in the combination of Eq. (3.13).

Using the function $\Gamma_1^{\alpha\alpha'}$ (3.13) we may calculate the statistical expectation for the time-dependent correlation of fluctuating microscopic quantities, in much the same way as the one particle distribution function allows us to calculate the statistical expectation of local microscopic quantities. Consider the microscopic dynamical function $b^\alpha(Y;\vec{x})$ defined on the 6 dimensional phase space $Y=(\vec{y},\vec{v})$, where we have allowed for the parametric dependence of b^α on the physical space $\vec{x}$. Statistical averaging over the Liouvillian distribution D_1 of this quantity, leads to the macroscopic function $B^\alpha(\vec{x},t)=\langle b^\alpha\rangle$ according to

$$B^\alpha(\vec{x},t) = n_\alpha\int dY\; b^\alpha(Y;\vec{x})f_1^\alpha(Y;t). \tag{3.16}$$

All the microscopic functions that we will deal with are local, in the sense that they are all of the form

$$b^\alpha(Y;\vec{x}) = \beta^\alpha(\vec{v})\delta(\vec{y}-\vec{x}). \tag{3.17}$$

An example is the local number density

$$n^\alpha(Y;\vec{x}) = \delta(\vec{y}-\vec{x}) \tag{3.18}$$

and another is the total average kinetic energy density

$$\mathcal{E}^\alpha = \frac{1}{2} m_\alpha v^2 \delta(\vec{y}-\vec{x}) \tag{3.19}$$

Substituting (3.17) into (3.16) gives

$$\begin{aligned} B^\alpha(\vec{x},t) &= n_\alpha \int dY\ \beta^\alpha(\vec{v})\delta(\vec{y}-\vec{x}) f_1^\alpha(\vec{y},\vec{v},t) \\ &= n_\alpha \int d\vec{v}\ \beta^\alpha(\vec{v}) f^\alpha(\vec{x},\vec{v},t) \end{aligned} \tag{3.20}$$

where the only non-trivial part of the averaging concerns the velocity space. In the same way, the correlations of the fluctuation of the microscopic quantity $\delta b^\alpha = b^\alpha - \langle b^\alpha \rangle$

$$\begin{aligned} &\langle \delta b^\alpha(\vec{x},t)\delta b^{\alpha'}(\vec{x}',t')\rangle = \\ &\quad n_\alpha n_{\alpha'} \int dY dY'\ \beta^\alpha(\vec{v})\beta^{\alpha'}(\vec{v}')\delta(\vec{y}-\vec{x})\delta(\vec{y}'-\vec{x}')\Gamma_1^{\alpha\alpha'}(Y,t;Y',t'), \\ &\quad = \int d\vec{v}d\vec{v}'\ \beta^\alpha(\vec{v})\beta^{\alpha'}(\vec{v}')\Gamma_1^{\alpha\alpha'}(\vec{x},\vec{v},t;\vec{x}',\vec{v}',t'). \end{aligned} \tag{3.21}$$

A particularly important example is the expectation for the correlation of the fluctuation in the microscopic density $\beta^\alpha = 1$

$$\langle \delta n^\alpha(\vec{x},t)\delta n^{\alpha'}(\vec{x}',t')\rangle = n_\alpha n_{\alpha'} \int d\vec{v}d\vec{v}'\ \Gamma_1^{\alpha\alpha'}(X,t;X',t') \tag{3.22}$$

due to its role in the scattering of laser light ($\alpha = \alpha' = e$).

3.1.3 Expansion in the plasma parameter

Rostoker was the first to develop an approximate procedure for calculating the moments of D_2 (3.14, 3.15) from the Liouville equation (3.6) as a power series in the

Discreteness Parameters	Fluid-like Parameters
$\{e_a,\ m_a,\ 1/n_a,\ T_a\}\to 0$	$\{n_a e_a,\ m_a n_a,\ n_a T_a\}\to$ const.

Table 3.1.1: In the limit the discreteness parameters go to zero, the plasma becomes a continuous fluid

plasma discreteness parameters, e_a the charge per particle, m_a the mass per particle, $1/n_a$ the average volume per particle and T_a the average kinetic energy per particle. These quantities are taken to be infinitesimal, and all of the same order.

One is to imagine taking a finite volume of plasma V, and chopping up each particle into finer and finer pieces in such a way as the discreteness parameters e_a, m_a, $1/n_a$, and T_a go to zero, but leave the continuous fluid-like properties of charge density $n_a e_a$, mass density $n_a m_a$, and average kinetic energy density $n_a T_a$ unchanged, see Table 3.1.1. This procedure retains all the collective medium-like properties of the plasma, since the Debye length, $\lambda_{Da} = \sqrt{T_a/4\pi n_a e_a^2}$, ion acoustic phase speed $c_s = \sqrt{ZT_e/m_i}$ and plasma frequency ω_{pa} are unaffected by the transformation. Discrete particle properties such as the collision frequency, vanish, $\nu \sim ne^4/\sqrt{m}T^{3/2}$. It is convenient to work with dimensionless parameters, and only one such family of parameters of the same order can be formed from the set e_a, m_a, $1/n_a$, T_a. The dimensionless parameter ε_p is such a parameter, and is referred to as either the "plasma parameter" or "*the* discreteness parameter". It can be identified with the inverse number of particles in a Debye sphere

$$\varepsilon_p = \frac{1}{N_D} = \left(\frac{4\pi n}{3}\right)^{-1} \lambda_{Da}^{-3}, \tag{3.23}$$

$\lambda_{Da} = \sqrt{4\pi e_a^2 n_a/T_a}$. Rostoker's procedure is relevant to plasmas that are weakly coupled. A plasma is said to be weakly coupled if the average Coulomb interaction energy per particle e^2/r_0, $r_0 = (4\pi n/3)^{-1/3}$ is smaller than the average kinetic energy

per particle, T

$$\Gamma = \frac{e^2}{r_0 T} \ll 1, \tag{3.24}$$

where we have introduced the Coulomb coupling parameter Γ [39]. The condition of weak coupling is sufficient to guarantee the smallness of the plasma parameter and hence the validity of Rostoker's method, because of the relation

$$\varepsilon_p = (3\Gamma)^{3/2}. \tag{3.25}$$

Using this method Rostoker has carried out a comprehensive treatment for plasmas in thermal equilibrium to the lowest non-trivial order.

3.1.4 Systematic BBGKY approach

Rostoker has defined a hierarchy of functions, of which Ω_1 and F_1 are the first members.

$$\Omega_s^{\alpha_1;\alpha_1\cdots\alpha_s}(X_1,t;X_1'\cdots X_s',t') = \frac{V^{s+1}}{N_{\alpha_1}}\int dX_2\cdots dX_N\, dX_{s+1}'\cdots dX_N'\, D_2(\{X_i\},t;\{X_i'\},t'), \tag{3.26}$$

and

$$F_s^{\alpha_1;\alpha_2\cdots\alpha_{s+1}}(X_1,t;X_2'\cdots X_{s+1}',t') = V^{s+1}\int dX_2\cdots dX_N\, dX_1' dX_{s+1}'\cdots dX_N'\, D_2(\{X_i\},t;\{X_i'\},t'). \tag{3.27}$$

We have also introduced the s-particle distribution function

$$f_s(X_1\cdots X_s;t) \equiv V^s\int D_1(X,t)dX_{s+1}\cdots dX_N. \tag{3.28}$$

The Equations satisfied by the Ω_s may be obtained from the Liouville equation (3.6) for D_2. Firstly, all the unprimed variables are integrated out with the exception of the "test particle" X_1. One is free to do this as the Liouville equation only acts on

the primed particles. Next one integrates out all but s of the primed particles. This procedure is exactly the same as for the single particle distribution function f_s. It is therefore no surprise that the Ω_s satisfy

$$\left(\frac{\partial}{\partial t}+\sum_{i=1}^{s}\vec{v}_i\cdot\frac{\partial}{\partial\vec{x}_i}+\vec{F}_{ex}(X_i,t)\cdot\frac{\partial}{\partial\vec{v}_i}\right.$$
$$\left.-\sum_{j\neq i}^{s}\frac{e_{\alpha_i}e_{\alpha_j}}{m_{\alpha_i}}\frac{\partial}{\partial\vec{x}_i}\frac{1}{|\vec{x}_i-\vec{x}_j|}\cdot\frac{\partial}{\partial\vec{v}_i}\right)\Omega_s^{\alpha_1;\alpha_1\cdots\alpha_s}(X_0,t_0;X_1\cdots X_s,t)=$$
$$+\sum_{i=1}^{s}\frac{e_{\alpha_i}}{m_{\alpha_i}}\sum_{r'}n_{r'}e_{r'}\int dX'\frac{\partial}{\partial\vec{x}_i}\frac{1}{|\vec{x}_i-\vec{x}'|}\cdot\frac{\partial}{\partial\vec{v}_i}\Omega_{s+1}^{\alpha_1;\alpha_1\cdots\alpha_s,r'}(X_0,t_0;X_1\cdots X_s,X',t). \tag{3.29}$$

From (3.29) we see that the test particle distribution changes due to interaction with the field particles. The equations for the field particle functions may be obtained from Eq. (3.29), by writing (3.29) for $s+1$ and then integrating out particle X_1.

$$\left(\frac{\partial}{\partial t}+\sum_{i=2}^{s+1}\vec{v}_i\cdot\frac{\partial}{\partial\vec{x}_i}+\vec{F}_{ex}(X_i,t)\cdot\frac{\partial}{\partial\vec{v}_i}\right.$$
$$\left.-\sum_{j\neq i}^{s+1}\frac{e_{\alpha_i}e_{\alpha_j}}{m_{\alpha_i}}\frac{\partial}{\partial\vec{x}_i}\frac{1}{|\vec{x}_i-\vec{x}_j|}\cdot\frac{\partial}{\partial\vec{v}_i}\right)F_s^{\alpha_1;\alpha_2\cdots\alpha_{s+1}}(X_0,t_0;X_1\cdots X_{s+1},t)=$$
$$n_{\alpha_1}e_{\alpha_1}\sum_{i=2}^{s+1}\frac{e_{\alpha_i}}{m_{\alpha_i}}\int dX_i\frac{\partial}{\partial\vec{x}_i}\frac{1}{|\vec{x}_i-\vec{x}_1|}\cdot\frac{\partial}{\partial\vec{v}_i}\Omega_{s+1}^{\alpha_1;\alpha_1\cdots\alpha_{s+1}}(X_0,t_0;X_1\cdots X_{s+1},t)$$
$$+\sum_{r'}n_{r'}e_{r'}\sum_{i=2}^{s+1}\frac{e_{\alpha_i}}{m_{\alpha_i}}\int dX'\frac{\partial}{\partial\vec{x}_i}\frac{1}{|\vec{x}_i-\vec{x}'|}\cdot\frac{\partial}{\partial\vec{v}_i}F_{s+1}^{\alpha_1;\alpha_2\cdots\alpha_{s+1},r'}(X_0,t_0;X_2\cdots X_{s+1},X',t). \tag{3.30}$$

Equation (3.30) says that the field particle distribution F_s is changed by interaction of field particles with both test particles and other field particles. The disturbance in the field particle distribution function δF_s, defined by

$$\delta F_s^{\alpha_1;\alpha_2\cdots\alpha_{s+1}}(X_1,t;X_2'\cdots X_{s+1}',t')=$$
$$F_s^{\alpha_1;\alpha_2\cdots\alpha_{s+1}}(X_1,t;X_2'\cdots X_{s+1}',t')-f_1^{\alpha_1}(X_1,t)f_s^{\alpha_2\cdots\alpha_{s+1}}(X_2'\cdots X_{s+1}',t') \tag{3.31}$$

will satisfy the same equation as the F's, as can easily be verified by substituting (3.31) into (3.30). The functions $\Gamma_s^{\alpha_0;\alpha_1\cdots\alpha_s}(X_0,t_0;X_1\cdots X_s,t)$ are defined in terms of

Ω and F in the following way

$$\begin{aligned}
&\Gamma_s^{\alpha_0;\alpha_1\cdots\alpha_s}(X_0,t_0;X_1\cdots X_s,t)\\
&\quad=[\delta F_s^{\alpha_0;\alpha_1\cdots\alpha_s}(X_0,t_0;X_1\cdots X_s)+\delta_{\alpha_0\alpha_1}\Omega_s^{\alpha_0;\alpha_1\cdots\alpha_s}(X_0,t_0;X_1\cdots X_s,t)\\
&\quad+\delta_{\alpha_0\alpha_2}\Omega_s^{\alpha_0;\alpha_2\alpha_1\cdots\alpha_s}(X_0,t_0;X_2,X_1,X_3\cdots X_s,t)+\cdots\\
&\quad+\delta_{\alpha_0\alpha_s}\Omega_s^{\alpha_0;\alpha_s\alpha_2\cdots\alpha_{s-1}\alpha_1}(X_0,t_0;X_s,X_2,X_3\cdots X_{s-1}X_1,t)],
\end{aligned}\tag{3.32}$$

which amounts to the identification of Γ_s as the probability of having a particle of species α_0 at the phase point X_0 at the time t_0 and then at the later times the particles at the phase points $X_1,\cdots X_s$, without regard for which particles were test particles or field particles. Because the identity of the test particles are lost the combination of Eq. (3.32) the Γ_s satisfy the BBGKY hierarchy

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+\sum_{i=1}^{s}\vec{v}_i\cdot\frac{\partial}{\partial\vec{x}_i}+\vec{F}_{ex}(X_i,t)\cdot\frac{\partial}{\partial\vec{v}_i}\right.\\
&\qquad\left.-\overbrace{\sum_{j\neq i}^{s}\frac{e_{\alpha_i}e_{\alpha_j}}{m_{\alpha_i}}\frac{\partial}{\partial\vec{x}_i}\frac{1}{|\vec{x}_i-\vec{x}_j|}\cdot\frac{\partial}{\partial\vec{v}_i}}^{a}\right)\Gamma_s^{\alpha_1;\alpha_1\cdots\alpha_s}(X_0,t_0;X_1\cdots X_s,t)=\\
&\qquad+\sum_{i=1}^{s}\frac{e_{\alpha_i}}{m_{\alpha_i}}\sum_{r'}n_{r'}e_{r'}\int dX'\frac{\partial}{\partial\vec{x}_i}\frac{1}{|\vec{x}_i-\vec{x}'|}\\
&\qquad\cdot\frac{\partial}{\partial\vec{v}_i}\Gamma_{s+1}^{\alpha_1;\alpha_1\cdots\alpha_s,r'}(X_0,t_0;X_1\cdots X_s,X',t).
\end{aligned}\tag{3.33}$$

This is the important result emphasized in the review by Oberman and Williams [6]. Although there is nothing essentially new in comparison with the earlier work of Rostoker [7] and also Hinton [18], the identification that the particular combination (3.32) obeys the BBGKY hierarchy means that standard existing techniques can be used to obtain a closed equation for Γ_1. Finding the approximate solutions for the Ω and F functions and then combining these solutions into the combination Γ_s is more complicated, although this was the method originally used by Rostoker [7].

The type of structure (3.33) is well known from plasma kinetic theory [10] where one attempts to obtain a closed equation from the BBGKY hierarchy for the

single particle distribution function f_1.

$$\left(\frac{\partial}{\partial t}+\sum_{i=1}^{s}\vec{v}_i\cdot\frac{\partial}{\partial\vec{x}_i}+\vec{F}_{ex}(X_i,t)\cdot\frac{\partial}{\partial\vec{v}_i}-\frac{e_{\alpha_i}}{m_{\alpha_i}}\sum_{j\neq i}^{s}\frac{\partial}{\partial\vec{x}_i}\frac{e_{\alpha_j}}{|\vec{x}_i-\vec{x}_j|}\cdot\frac{\partial}{\partial\vec{v}_i}\right)f_s=$$
$$-\sum_{i=1}^{s}\frac{nq^2}{m}\int\frac{\partial}{\partial\vec{x}_i}\frac{1}{|\vec{x}_i-\vec{x}_{s+1}|}\cdot\frac{\partial f_{s+1}}{\partial\vec{v}_i}dX_{s+1}. \tag{3.34}$$

In Eq. (3.34) the s particle distribution function f_s does not satisfy the Liouville equation (3.6) because of the interaction with the $(s+1)$ distribution function, f_{s+1}. In the plasma regime $\varepsilon_p \ll 1$, the BBGKY hierarchy is truncated by a correlation expansion.

3.1.5 Correlation expansion

To the lowest order in ε_p we neglect term a in Eq. (3.33) as it is of higher order in the plasma parameter (c.f. Table 3.1.1), take $f_s^{\alpha_1\cdots\alpha_s}(X_1,\cdots X_s)=f_1^{\alpha_1}(X_1,t)\cdots f_1^{\alpha_s}(X_s,t)$, and a linearized version for the Γ's,

$$\begin{aligned}
&\Gamma_s^{\alpha_0;\alpha_1\cdots\alpha_s}(X_0,t_0;X_1\cdots X_s,t)=\\
&\quad\Gamma_1^{\alpha_0;\alpha_1}(X_0,t_0;X_1,t)f_1^{\alpha_2}(X_2,t)\cdots f_1^{\alpha_s}(X_s,t)\\
&\quad+f_1^{\alpha_1}(X_1,t)\Gamma_1^{\alpha_0;\alpha_2}(X_0,t_0;X_2,t)f_1^{\alpha_3}(X_3,t)\cdots f_1^{\alpha_s}(X_s,t)+\cdots\\
&\quad+f_1^{\alpha_1}(X_1,t)\cdots f_1^{\alpha_{s-1}}(X_{s-1},t)\Gamma_1^{\alpha_0;\alpha_s}(X_0,t_0;X_s,t).
\end{aligned}\tag{3.35}$$

This solves (3.33) as long as the single particle distribution function f_1^α satisfies the Vlasov equation,

$$\frac{\partial}{\partial t}f^\alpha(X,t)+\vec{v}\cdot\frac{\partial}{\partial\vec{x}}f^\alpha(X,t)-\frac{e_\alpha}{m_\alpha}\vec{E}\cdot\frac{\partial}{\partial\vec{v}}f^\alpha(X,t)=0, \tag{3.36}$$

$$\nabla\cdot\vec{E}=4\pi\sum_\beta e_\beta\int d\vec{v}\,f^\beta(\vec{x},\vec{v},t), \tag{3.37}$$

and Γ_1 satisfies the linearized Vlasov equation, as can be verified by direct substitution of (3.35) into (3.33). It is this observation that motivates the correlation expansion.

For our purposes we will need to introduce a few of the lower order irreducible functions in the cumulant expansion[1] for the s-particle distribution function, f_s, and the linearized version for the Γ_s's. Cumulant functions will be denoted with a bar.

$$f_1(X_1,t) = \bar{f}_1(X_1,t) \tag{3.38}$$

$$f_2(X_1,X_2,t) = \varepsilon_p \bar{f}_2(X_1,X_2,t) + \bar{f}_1(X_1,t)\bar{f}_1(X_2,t) \tag{3.39}$$

and for the Γ's

$$\Gamma_1(X_0,t_0;X_1,t) = \varepsilon_p \bar{\Gamma}_1(X_0,t_0;X_1,t) \tag{3.40}$$

$$\begin{aligned}\Gamma_2(X_0,t_0;X_1,X_2,t) &= \varepsilon_p^2 \bar{\Gamma}_2(X_0,t_0;X_1,X_2,t) + \varepsilon_p \bar{\Gamma}_1(X_0,t_0;X_1,t)\bar{f}_1(X_2,t) \\ &+ \varepsilon_p \bar{f}_1(X_1,t)\bar{\Gamma}_1(X_0,t_0;X_2,t)\end{aligned} \tag{3.41}$$

$$\begin{aligned}\Gamma_3(X_0,t_0;X_1,X_2,X_3,t) &= \varepsilon_p^3 \bar{\Gamma}_3(X_0,t_0;X_1,X_2,X_3,t) + \varepsilon_p^2 \bar{\Gamma}_2(X_0,t_0;X_1,X_2,t)\bar{f}_1(X_3,t) \\ &+ \varepsilon_p^2 \bar{\Gamma}_2(X_0,t_0;X_2,X_3,t)\varepsilon_p^2 \bar{f}_1(X_1,t) + \varepsilon_p^2 \bar{\Gamma}_2(X_0,t_0;X_3,X_1,t)\bar{f}_1(X_2,t) \\ &+ \varepsilon_p^2 \bar{f}_2(X_1,X_2,t)\bar{\Gamma}_1(X_0,t_0;X_3,t) + \varepsilon_p^2 \bar{f}_2(X_2,X_3,t)\bar{\Gamma}_1(X_0,t_0;X_1,t) \\ &+ \varepsilon_p^2 \bar{f}_2(X_3,X_1,t)\bar{\Gamma}_1(X_0,t_0;X_2,t) + \varepsilon_p \bar{\Gamma}_1(X_0,t_0;X_1,t)\bar{f}_1(X_2,t)\bar{f}_1(X_3,t) \\ &+ \varepsilon_p \bar{f}_1(X_1,t)\bar{\Gamma}_1(X_0,t_0;X_2,t)\bar{f}_1(X_3,t) \\ &+ \varepsilon_p \bar{f}_1(X_1,t)\bar{f}_1(X_2,t)\bar{\Gamma}_1(X_0,t_0;X_3,t)\end{aligned} \tag{3.42}$$

The lower order irreducible correlation functions are usually given names. We will name them according to

$$\bar{f}_1(X_1,t) = f_1(X_1,t) \tag{3.43}$$

$$\bar{f}_2(X_1,X_2,t) = g(X_1,X_2,t) \tag{3.44}$$

and for the two-time quantities

$$\bar{\Gamma}_1(X_0,t_0;X_1,t) = \Gamma_1(X_0,t_0;X_1,t) \tag{3.45}$$

$$\bar{\Gamma}_2(X_0,t_0;X_1,X_2,t) = \Delta(X_0,t_0;X_1,X_2,t) \tag{3.46}$$

$$\bar{\Gamma}_3(X_0,t_0;X_1,X_2,X_3,t) = \Upsilon(X_0,t_0;X_1,X_2,X_3,t). \tag{3.47}$$

[1]The cumulant expansion is also known as a Mayer cluster expansion [40].

It is now a matter of substituting the definitions (3.38-3.47) into the BBGKY hierarchy for Γ_s (3.33), in order to derive the equations satisfied by the cumulant functions to a given order in the plasma parameter, ε_p.

3.1.6 Results to lowest order in ε_p

From Sec. 3.1.5 we have seen Γ_1 obeys the linearized Vlasov equation to lowest order in the plasma parameter

$$\left(\frac{\partial}{\partial t}+\vec{v}\cdot\frac{\partial}{\partial \vec{x}}+\frac{e_\alpha}{m_\alpha}\vec{E}_0(\vec{x},t)\cdot\frac{\partial}{\partial \vec{v}}\right)\Gamma_1^{\alpha_0;\alpha}(X_0,t_0;X,t)$$
$$-\frac{e_\alpha}{m_\alpha}\frac{\partial f(X,t)}{\partial \vec{v}}\cdot\sum_{\alpha'}n_{\alpha'}e_{\alpha'}\int dX'\,\Gamma_1^{\alpha_0;\alpha'}(X_0,t_0;X',t)\cdot\frac{\partial}{\partial \vec{x}}\frac{1}{\mid \vec{x}-\vec{x}'\mid}=0. \quad (3.48)$$

The initial condition is,

$$\frac{1}{n_{\alpha_0}}\Gamma_1^{\alpha_0;\alpha}(X_0,t_0;X,t_0)=\delta_{\alpha_0,\alpha}\delta(X_0-X)f^{\alpha_0}(X_0,t_0)+g^{\alpha_0\alpha}(X_0,X,t_0). \quad (3.49)$$

Where $g(X',X,t)$ is the irreducible part of the two-body correlation function (3.43), and we have allowed for the presence of a background electric field $\vec{E}_0$. At time t_0 the plasma is caught in the correlated state given by Eq. (3.49) which introduces the discreteness. A Vlasov fluctuation then propagates to the time t when the second measurement of the plasma is made. Hence, to this order in ε_p the fluctuations are not collisionally damped. From Eqns. (3.48-3.49) all of Rostoker's results [7] including $S(\vec{k},\omega)$ (2.18) that we have seen in Section 2.2, can be reproduced [6].

3.1.7 Results to order ε^2

To the next order $O(\varepsilon_p^2)$ Eq. (3.48) is modified by the addition of a "collision term" on the right hand side,

$$\frac{e_\alpha}{m_\alpha}\sum_{\alpha'}n_{\alpha'}e_{\alpha'}\int dX'\,\frac{\partial}{\partial \vec{x}}\frac{1}{\mid \vec{x}-\vec{x}'\mid}\cdot\frac{\partial}{\partial \vec{v}}\Delta^{\alpha_0;\alpha\alpha'}(X_0,t_0;X,X',t). \quad (3.50)$$

In order to evaluate this collision integral, it is necessary to solve the equation for $\Delta(X_0,t_0;X,X',t)$. This equation is found to be [6]

$$\left(\frac{\partial}{\partial t}+\vec{v}\cdot\frac{\partial}{\partial\vec{x}}+\frac{e_\alpha}{m_\alpha}\vec{E}_0\cdot\frac{\partial}{\partial\vec{v}}+\vec{v}'\cdot\frac{\partial}{\partial\vec{x}'}+\frac{e_{\alpha'}}{m_{\alpha'}}\vec{E}_0\cdot\frac{\partial}{\partial\vec{v}'}\right)\Delta^{\alpha_0;\alpha\alpha'}(X_0,t_0;X,X',t)=$$
$$\vec{A}\left[\Gamma_1^{\alpha_0;\alpha}(X_0,t_0;X,t)f(X',t)+\Gamma_1^{\alpha_0;\alpha'}(X_0,t_0;X',t)f(X,t)\right]$$
$$+\left\{\sum_{\alpha''}m_{\alpha''}e_{\alpha''}\int dX''\frac{e_\alpha}{m_\alpha}\frac{\partial}{\partial\vec{x}}\frac{1}{|\vec{x}-\vec{x}''|}\cdot\frac{\partial}{\partial\vec{v}}\Big[\right.$$
$$\overbrace{f_1(X,t)\Delta^{\alpha_0;\alpha'\alpha''}(X_0,t_0;X',X'',t)}^{a}+\overbrace{\Gamma^{\alpha_0;\alpha''}(X_0,t_0;X'',t)g(X,X',t)}^{b}$$
$$\left.+\overbrace{\Gamma^{\alpha_0;\alpha}(X_0,t_0;X,t)g(X',X'',t)}^{b}\Big]+(X\leftrightarrow X')\right\}. \tag{3.51}$$

where

$$\vec{A}=e_\alpha e_{\alpha'}\frac{\partial}{\partial\vec{x}}\frac{1}{|\vec{x}-\vec{x}'|}\cdot\left(\frac{1}{m_\alpha}\frac{\partial}{\partial\vec{v}}-\frac{1}{m_{\alpha'}}\frac{\partial}{\partial\vec{v}'}\right). \tag{3.52}$$

This equation is the exact linearization of the equation for the two body correlation function $g(X,X',t)$ that one is usually concerned with in plasma kinetic theory [41]. i.e. if the replacement

$$g \longrightarrow g+\Delta$$
$$f \longrightarrow f+\Gamma$$

is made in the equation for $g(X,X',t)$ and linearized treating Γ and Δ as small quantities, then the result is (3.51). We can then appeal to the same method as used in the one-time case for the solution of $g[f_1]$ in terms of the one-particle distribution function. There are some subtleties involved in this solution, but these are well treated in many text books, cf. e.g. [41].

In the one-time case for f_1 the equivalent expression to (3.50) leads to the Balescu-Guernsey-Lenard (BGL) collision term [41]. It seems plausible that the term (3.50) will therefore be the linearized BGL collision term, and this has in fact been proved by Krommes [83].

As a formal procedure for calculating Γ_1 we will write the linearized BGL equation for $\delta f^\alpha(X,t)$ (3.8)

$$\left(\frac{\partial}{\partial t}+\vec{v}\cdot\frac{\partial}{\partial \vec{x}}\right)\delta f^\alpha-\frac{e_\alpha}{m_\alpha}\frac{\partial\langle f^\alpha\rangle}{\partial\vec{v}}\cdot\frac{\partial}{\partial\vec{x}}\delta\phi=\sum_\beta C[\delta f^\alpha,\langle f^\beta\rangle]+C[\langle f^\alpha\rangle,\delta f^\beta]. \quad (3.53)$$

where

$$\delta\phi(\vec{x},\vec{v},t)=\sum_\beta e_\beta\int d\vec{x}'d\vec{v}'\frac{\delta f^\beta(\vec{x}',\vec{v}',t)}{\mid\vec{x}-\vec{x}'\mid}. \quad (3.54)$$

with the initial condition $\delta f^\alpha(\vec{k},\vec{v},t=0)$. The Fourier-Laplace transform of the function Γ_1 is obtained by multiplying the solution $\delta f^\alpha(\vec{k},\vec{v},p)$ by $\delta f^{\beta*}(\vec{k},\vec{v}',t=0)$ and ensemble averaging in order to obtain $\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',p)=\langle\delta f^\alpha(\vec{k},\vec{v},p)\delta f^{\beta*}(\vec{k},\vec{v}',0)\rangle$ in terms of the initial correlations. $\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',p)$ is the Laplace transform of $\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',t)$ $=\langle\delta f^\alpha(\vec{k},\vec{v},t)\delta f^{\beta*}(\vec{k},\vec{v}',0)\rangle$,

$$\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',p)=\int_0^\infty dt\exp(ipt)\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',t). \quad (3.55)$$

Due to the time reversal symmetry of $\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',t)$ and the fact it is a real quantity, its Fourier transform can be written in terms of its Laplace transform (3.55) according to

$$\begin{aligned}\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',\omega)&=\int_0^\infty dt\exp(i\omega t)\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',t)+\\&\qquad\int_{-\infty}^0 dt\exp(i\omega t)\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',t)\\&=\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',p)+\left[\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',p)\right]^*\\&=2\mathrm{Re}\,\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',p). \quad (3.56)\end{aligned}$$

$\Gamma^{\alpha\beta}(\vec{k},\vec{v},\vec{v}',\omega)$ (3.56) may be used in order to obtain spectral functions of macroscopic quantities (3.21). Of particular importance is the dynamical form factor $S(\vec{k},\omega)$ (3.22),

$$S(\vec{k},\omega)=n_e\int d\vec{v}\,d\vec{v}'\Gamma^{ee}(\vec{k},\vec{v},\vec{v}',\omega)=2\mathrm{Re}\frac{\langle\delta n_e(\vec{k},\omega)\delta n_e^*(\vec{k},0)\rangle}{n_e}. \quad (3.57)$$

Hydrodynamic fluctuations

The BGL theory (3.53) includes *collisional* damping of the fluctuations, in addition to the Landau damping of the Vlasov theory (3.48). Eq. (3.53) is appropriate for a discussion of fluctuations on both kinetic and hydrodynamic scales. In fact, the BGL collision operator, which in turn can be approximated by the Landau collision term (by neglecting the screening terms a, and b in Eq. (3.51)) allows for the existence of hydrodynamic modes [83].

In the hydrodynamic regime Eq. (3.53) can be solved by a modification of the usual Chapman-Enskog [1] method, resulting in a system of linear fluid equations for the fluctuating hydrodynamic quantities, $\{\delta n_\alpha, \delta\vec{u}_\alpha, \delta T_\alpha\}$. These may be obtained by linearizing the usual Braginskii fluid equations

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial \vec{x}}(n_\alpha \vec{u}_\alpha) = 0, \tag{3.58}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}_\alpha \cdot \frac{\partial}{\partial \vec{x}}\right)\vec{u}_\alpha = -\frac{1}{m_\alpha n_\alpha}\frac{\partial}{\partial \vec{x}}(n_\alpha T_\alpha) - \frac{1}{m_\alpha n_\alpha}\frac{\partial}{\partial \vec{x}}\cdot\hat{\sigma}_\alpha - \frac{e_\alpha}{m_\alpha}\frac{\partial}{\partial \vec{x}}\phi + \frac{1}{m_\alpha n_\alpha}\vec{R}_\alpha, \tag{3.59}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}_\alpha \cdot \frac{\partial}{\partial \vec{x}}\right)T_\alpha + \frac{2T_\alpha}{3}\frac{\partial}{\partial \vec{x}}\cdot\vec{u}_\alpha = - \frac{2}{3n_\alpha}\frac{\partial}{\partial \vec{x}}\cdot\vec{q}_\alpha - \frac{2}{3n_\alpha}\hat{\sigma}_\alpha \cdot \frac{\partial \vec{u}_\alpha}{\partial \vec{x}} + \frac{2}{3n_\alpha}Q_\alpha, \tag{3.60}$$

with $n_\alpha \to n_\alpha + \delta n_\alpha$, and so on [6]. The linearization of equations (3.58)-(3.60) together with the linearized closure relations that relate the fluctuating fluxes $\{\delta\vec{q}_\alpha, \delta\hat{\sigma}_\alpha, \delta\vec{R}, \delta Q\}$ to the forces $\{-\nabla\delta T_\alpha, \delta\hat{W}, \delta\vec{u}\}$ as a result of the Chapman-Enskog procedure [1] will form a complete set of equations from which one can calculate the thermal correlations of any of the hydrodynamic variables, for example $\langle \delta n_\alpha \delta n_\beta^* \rangle / n_e$ in terms of the initial correlations. The initial correlations for a weakly coupled equilibrium plasma can be obtained by taking the appropriate velocity moments of the initial

condition on $\Gamma^{\alpha\beta}$ (3.49), given by

$$\langle \delta f^{\alpha}(\vec{x},\vec{v},0)\delta f^{\beta}(\vec{x}',\vec{v}',0)\rangle = \delta_{\alpha\beta}\delta(\vec{v}-\vec{v}')\delta(\vec{x}-\vec{x}')f^{\alpha}_{M}(v)/n_{\alpha}, \tag{3.61}$$

where $f^{\alpha}_{M}(v)$ is a Maxwellian distribution function,

$$f^{\alpha}_{M}(v) = 1/(\sqrt{2\pi}v_{T\alpha})^{3}\exp\left(-v^{2}/2v^{2}_{T\alpha}\right), \tag{3.62}$$

and $v_{T\alpha} = \sqrt{T_{\alpha}/m_{\alpha}}$ is the thermal velocity of particles of species α.

We see that this is the Onsager method described in Chapter 2 in the context of neutral fluids, although here it has been shown to be the consequence of the kinetic equation (3.53) for the phase space fluctuation δf. Furthermore, the validity of the description in terms of the hydrodynamic variables is given by the validity conditions of the Chapman-Enskog method, which is something that we will now explore.

Chapter 4

FLUID DESCRIPTION OF PLASMAS

4.1 Hydrodynamic reduction

In a hydrodynamic description the interest is turned away from the evolution of the one particle distribution function $f^\alpha(\vec{x},\vec{v},t)$ towards its first five velocity moments,

$$n^\alpha(\vec{x},t) = \int d\vec{v}\, f^\alpha(\vec{x},\vec{v},t), \tag{4.1}$$

$$n^\alpha(\vec{x},t)\vec{u}^\alpha(\vec{x},t) = \int d\vec{v}\,\vec{v} f^\alpha(\vec{x},\vec{v},t), \tag{4.2}$$

$$n^\alpha(\vec{x},t)T^\alpha(\vec{x},t) = \frac{m_\alpha}{3}\int d\vec{v}\mid \vec{v}-\vec{u}^\alpha(\vec{x},t)\mid^2 f^\alpha(\vec{x},\vec{v},t). \tag{4.3}$$

In spite of this loss of information, the hydrodynamic equations provide an excellent description of plasma dynamics over a wide range of parameters. Therefore solutions of the kinetic theory should be reducible to a fluid description in the cases where such a fluid theory is expected to be valid. Investigations into the reduction of the kinetic description in terms of hydrodynamic equations has led to many advances in the field of nonequilibrium statistical mechanics, some of which we will now examine. Many of these early investigations centered around the Boltzmann equation [61], which is a nonlinear, integro-differential equation describing the evolution of the distribution function for a low density gas of classical particles, whose collisions may be considered as separate binary events,

$$\frac{\partial}{\partial t}f^\alpha + \vec{v}\cdot\nabla f^\alpha = \sum_\beta J(f^\alpha,f^\beta),$$

$$J(f^\alpha,f^\beta) = \int (f'^\alpha(\vec{v}')f'^\beta(\vec{v}_1{}') - f^\alpha(\vec{v})f^\beta(\vec{v}_1))\, gb\, db\, d\epsilon\, d\vec{v}_1. \tag{4.4}$$

Here $g = \mid \vec{v}_1 - \vec{v} \mid$ is the relative speed between the two colliding particles, b is the impact parameter, and ϵ is the angle between the orbital plane and the plane containing the velocities of the two particles [47]. The f'^{α} and f'^{β} are the values of f^{α} and f^{β} for velocities $(\vec{v}', \vec{v}_1{}')$ such that a particle of type α will be left after the collision with a velocity in the phase element, $d\vec{v}$ about the point $\vec{v}$. The other product $f^{\alpha} f^{\beta}$ in the collision term (4.4) represents a loss of particles of type α from the same phase element. Many fundamental properties have been proven for the Boltzmann equation including the irreversibility of the approach of the one particle distribution function towards equilibrium expressed by the H-theorem,

$$f^{\alpha}(\vec{x}, \vec{v}, t) \to n(\vec{x}, t) \left(\frac{m_{\alpha}}{2\pi T(\vec{x}, t)} \right)^{3/2} \exp\left(\frac{m_{\alpha}(\vec{v} - \vec{u}(\vec{x}, t))^2}{2T(\vec{x}, t)} \right). \tag{4.5}$$

For small deviations from the local equilibrium Maxwellian distribution function (4.5) one can prove existence and uniqueness theorems for the linearized collision operator. Properties of the general collision operator include the existence of the collisional invariants

$$\int d\vec{v}\, J(f, f) = 0, \quad \int d\vec{v}\, \vec{v}\, J(f, f) = 0, \quad \text{and} \int d\vec{v}\, v^2\, J(f, f) = 0, \tag{4.6}$$

that are consequences of the conservation of number density, momentum and energy. Taking the moments implied by (4.6) of the kinetic equation (4.4) generates the equations of continuity, momentum transfer and temperature,

$$\frac{\partial}{\partial t} n + \nabla \cdot (n\vec{u}) = 0 \tag{4.7}$$

$$m \frac{\partial}{\partial t}(n\vec{u}) + \nabla \cdot (\hat{P} + mn\vec{u}\vec{u}) = 0 \tag{4.8}$$

$$\frac{\partial}{\partial t} n \left(\frac{3}{2} T + \frac{m}{2} u^2 \right) + \nabla \cdot \left\{ \vec{q} + n \left(\frac{3}{2} T + \frac{m}{2} u^2 \right) \vec{u} + \hat{P} \cdot \vec{u} \right\} = 0. \tag{4.9}$$

Because of the conditions (4.6), collisions do not directly effect the moments (4.1-4.3), and hence we will sometimes refer to them as the *conserved* moments. The conserved moments are only effected by collisions through the higher order moments,

the pressure tensor $\hat{P}$, and heat flux $\vec{q}$ that appear due to the term $\vec{v} \cdot \partial f / \partial \vec{x}$ in the kinetic equation. These equations (4.7-4.9) are not closed as the pressure tensor

$$\hat{P} = m \int d\vec{v} (\vec{v} - \vec{u})(\vec{v} - \vec{u}) f(\vec{x}, \vec{v}, t) \tag{4.10}$$

and heat flux

$$\vec{q} = \frac{m}{2} \int d\vec{v} \mid \vec{v} - \vec{u} \mid^2 (\vec{v} - \vec{u}) f(\vec{x}, \vec{v}, t) \tag{4.11}$$

are unknown moments of the solution to the kinetic equation. By including equations for these moments (4.10) and (4.11) one does not resolve the closure problem as each new equation will always be coupled to a higher order moment. A solution to the closure problem involves expressing the higher order moments (4.10) and (4.11) in terms of the conserved moments (4.1-4.3) and their gradients, in a similar way as for example the phenomenological Fourier law relates heat flux to gradients in temperature [41].

Before discussing methods of solution to the Boltzmann equation and the closure problem it is necessary to examine the remaining terms in (4.4). The different terms in the Boltzmann equation (4.4) introduce the characteristic scales τ_H, and L_H that come from the free flow terms on the lhs of (4.4) and are the temporal and spatial scales over which the macroscopic quantities (4.1-4.3) vary. The collision term introduces the collision time τ_a, that in the absence of any spatial gradients is the time over which the distribution function relaxes to the equilibrium Maxwellian (4.5). One expects hydrodynamics to be valid when the kinetic equation is dominated by the collision term and the conserved moments (4.1-4.3) are essentially constant over the scale on which collisions occur. This will be the case when the following ordering is satisfied

$$\max\{l_{a\beta}\} \ll L_H, \tag{4.12}$$

$$\tau_a \ll \tau_H, \tag{4.13}$$

where $l_{\alpha\beta}$ is the collsional mean free path for species α colliding with species β. The parameter

$$\delta = \max\{l_{\alpha\beta}/L_H, \tau_\alpha/\tau_H\} \tag{4.14}$$

is a measure of importance of the collision term and is small in the regime given by (4.12, 4.13), $\delta \ll 1$. Hilbert has proved a theorem with regards to the Boltzmann equation that says, if the equation is dominated by collisions i.e. the collision term is weighted by $1/\delta$, and the distribution function can be expanded in a power series in δ about $\delta = 0$, then we can extract a macroscopic description in terms of density, velocity and temperature [62, 63], resolving the closure problem.

Hilbert's theorem 4.1 *If the kinetic equation can be written*

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = \frac{1}{\delta} J(f, f) \tag{4.15}$$

and if

$$f = \sum_{n=0}^{\infty} \delta^n f^{(n)} \tag{4.16}$$

then the solution for f is unique and determined by its five moments at the time $t = 0$.

$$n(\vec{x}, 0), \quad \vec{u}(\vec{x}, 0), \quad \textit{and} \quad T(\vec{x}, 0). \tag{4.17}$$

A proof of this theorem may be found in the references [61] and [45]. In the class of solutions to Boltzmann's equation that can be represented as a power series in δ (4.16) there is a one-one correspondence between the solutions $f(\vec{x}, \vec{v}, t)$ and the value of the conserved moments (4.1-4.3) at any "initial time". This class of solutions is referred to as either the "Hilbert Class" or the class of "normal solutions". Furthermore, this is true for any time and hence the Boltzmann equation is a one-one mapping of the Hilbert class into itself. Hilbert has proved that the density, hydrodynamic velocity, and temperature can be determined directly from their values at the initial time. Also, since the one-one correspondence between the distribution function f and its

conserved moments is preserved in time, the pressure (4.10) and heat flux (4.11) may be determined at any time directly in terms of n, $\vec{v}$ and T. Substituting these into the closure relations (4.10) and (4.11) will thus solve the closure problem.

We briefly review the Hilbert method (cf. e.g. [61]) On substituting the expansion for the distribution function (4.16) into the kinetic equation (4.15) one obtains a set of equations for each order in δ. To the lowest order the equation to be solved is simply

$$J(f^{(0)}, f^{(0)}) = 0. \tag{4.18}$$

This has the local Maxwellian distribution function (4.5) as a solution, with the five free parameters that are the conserved moments, $n^{(0)}$, $\vec{v}^{(0)}$, $T^{(0)}$. The next order approximation is the important one, and illustrates the essential steps that are carried out for each higher order in turn,

$$J(f^{(0)}, f^{(1)}) + J(f^{(1)}, f^{(0)}) = \mathcal{D}f^{(0)} \tag{4.19}$$

and generally

$$\begin{aligned} &J(f^{(0)}, f^{(r)}) + J(f^{(r)}, f^{(0)}) = \\ &\quad \mathcal{D}f^{(r-1)} - J(f^{(1)}, f^{(r-1)}) - \ldots - J(f^{(r-1)}, f^{(1)}), \end{aligned} \tag{4.20}$$

where the operator $\mathcal{D}$ represents the free flow term on the lhs of the Boltzmann equation (4.4). Hilbert has shown that (4.20) is an equation for $f^{(r)}/f^{(0)}$ of a particular type, a linear, inhomogeneous, Fredholm integral equation of the second kind [63]. The solution to such an equation consists of a particular solution, plus any linear combination of the independent solutions to the homogeneous equation, $f^{(r)}/f^{(0)} = \{1, \vec{v}, v^2\}$, i.e. the summational invariants of the collision operator. For this class of equations, a solubility condition exists requiring the inhomogeneous term on the rhs of (4.20) to be orthogonal to all the solutions to the homogeneous equation

[63]. These solubility conditions take the form

$$\int d\vec{v}\, \mathcal{D}f^{(r-1)} = 0, \quad \int d\vec{v}\, \vec{v}\mathcal{D}f^{(r-1)} = 0, \quad \int d\vec{v}\, v^2\mathcal{D}f^{(r-1)} = 0. \tag{4.21}$$

because the collision terms vanish by virtue of (4.6). These conditions (4.21) force the previously unspecified parameters in $f^{(r-1)}$ from the previous order to obey the ideal fluid (Euler) equations. To reiterate, the solubility condition for the $r = 1$ equation is such to fix the parameters $n^{(0)}$, $\vec{u}^{(0)}$ and $T^{(0)}$ in the $r = 0$ solution, by requiring them to satisfy the Euler equations. Furthermore, the zeroth order solution is known for all times if the initial values of $n^{(0)}$, $\vec{u}^{(0)}$ and $T^{(0)}$ are known, since the Euler equations are of first order in time. This procedure can be carried out to any order, and the unknown parameters at the r^{th} level are fixed at the $r+1$ level and each is uniquely determined on specification of the initial conditions, $n^{(r)}$, $\vec{u}^{(r)}$ and $T^{(r)}$, which are defined according to

$$n^{(r)} = \int d\vec{v}\, f^{(r)}, \quad n^{(r)}\vec{u}^{(r)} = \int d\vec{v}\, \vec{v} f^{(r)}, \quad \text{and} \quad n^{(r)}T^{(r)} = \frac{m}{3}\int d\vec{v}\, v^2 f^{(r)}. \tag{4.22}$$

In this way it can be seen that a specification of the full hydrodynamic moments

$$n = \sum_{r=0}^{\infty} n^{(r)}, \quad n\vec{u} = \sum_{r=0}^{\infty} n^{(r)}\vec{u}^{(r)}, \quad \text{and,} \quad nT = \sum_{r=0}^{\infty} n^{(r)}T^{(r)}, \tag{4.23}$$

at the initial time will determine uniquely the full solution $f = \sum_{r=0}^{\infty} f^{(r)}$ at any later time, assuming of course that the series for f converges. Such solutions are said to be members of the Hilbert class, or the class of *normal* solutions. The most serious problem with the Hilbert expansion is the inability of the method to generate the Navier Stokes equations of real fluid dynamics. The value of Hilbert's method is the introduction of normal solutions that are central to the Chapman-Enskog method, [48].

In the Chapman-Enskog method it is the special treatment of the time derivative that allows the method to generate the real fluid equations, and hence become

a practical procedure for the evaluation of transport coefficients. In order to achieve this closure it is *postulated* that time does not enter explicitly among the arguments of f and instead only comes through the implicit dependence of the conserved moments,

$$f(\vec{x},\vec{v},t) = f(\vec{x},\vec{v} \mid \vec{\beta}, \nabla\vec{\beta}\ldots), \tag{4.24}$$

$$\frac{\partial}{\partial t}\vec{\beta}(\vec{x},t) = \mathcal{M}(\vec{x} \mid \vec{\beta}, \nabla\vec{\beta}\ldots), \tag{4.25}$$

where $\vec{\beta}$ is defined as a vector whose components are the conserved moments, i.e. we have $\vec{\beta} = \{n(\vec{x},t), \vec{u}(\vec{x},t), T(\vec{x},t)\}$. Equation (4.25) implies the closed fluid description where the conserved moments only depend upon themselves and their gradients. As in the Hilbert method, the distribution function is expanded in δ, which implies through (4.25) that $\mathcal{M}$ itself is also expanded,

$$f = f^{(0)} + \delta f^{(1)} + \delta^2 f^{(2)} + \ldots \tag{4.26}$$

and

$$\mathcal{M} = \mathcal{M}^{(0)} + \delta\mathcal{M}^{(1)} + \delta^2\mathcal{M}^{(2)} + \ldots \tag{4.27}$$

Upon application of the chain rule and (4.25), the derivative of f reads

$$\frac{\partial f}{\partial t} = \mathcal{M}\cdot\nabla_\beta f + (\nabla_{\vec{x}}\mathcal{M}) : (\nabla_{\nabla_{\vec{x}}\beta}) f + \ldots \tag{4.28}$$

On substituting the expansions (4.26) and (4.27) into the above (4.28) gives

$$\frac{\partial f}{\partial t} = \frac{\partial_0 f^{(0)}}{\partial t} + \delta\left\{\frac{\partial_1 f^{(0)}}{\partial t} + \frac{\partial_0 f^{(1)}}{\partial t}\right\} + \ldots, \tag{4.29}$$

where the operator $\partial_i/\partial t$ is defined,

$$\frac{\partial_i}{\partial t} = \mathcal{M}^{(i)}\cdot\nabla_\beta f + (\nabla_{\vec{x}}\mathcal{M}^{(i)}) : (\nabla_{\nabla_{\vec{x}}\beta}) f + \ldots \tag{4.30}$$

The procedure now follows the Hilbert method, until the question of the solubility conditions are reached. Because of the manner in which the time derivative has been split up, the solubility conditions can now be achieved by requiring that $f^{(0)}$ alone

satisfies the ideal fluid equations (obtained from (4.7-4.9) on neglecting the traceless part of $\hat{P}$ and the heat flux $\vec{q}$) and determines completely the full conserved moments,

$$n = \int d\vec{v}\, f^{(0)}, \quad n\vec{u} = \int d\vec{v}\, \vec{v} f^{(0)}, \quad \text{and} \quad nT = \frac{m}{3} \int d\vec{v}\, v^2 f^{(0)}. \tag{4.31}$$

The higher order corrections $f^{(r)}$, $r > 0$ make no contribution to the conserved moments at all, but the $f^{(r)}$ are however determined in terms of the full n, $\vec{v}$, and T. The r^{th} order corrections to the pressure tensor (4.10) and heat flow (4.11) evaluated from $f^{(r)}$ will also be in terms of n, $\vec{v}$ and T and their spatial gradients. In this way a sequence of successively higher order fluid dynamic equations for n, $\vec{v}$ and T is obtained.

The distinction between the Hilbert and Chapman-Enskog methods is that in the Hilbert method the distribution function is expanded in a power series, whilst in the Chapman-Enskog method both the function and the equations are expanded. In this way it is possible to obtain the real fluid equations (Navier–Stokes) from the Chapman- Enskog method, but not from the Hilbert method (Which only generates the ideal fluid or Euler equations). In the Chapman-Enskog method, the transport coefficients of a gas may be obtained to any degree of accuracy by solving sets of linear algebraic equations, that result from the linear integral equation on expansion of the distribution function in a complete set of orthogonal polynomials.

So far the discussion has been focussed on the Boltzmann equation and neutral gases. Plasmas are quite distinct from neutral gases due to the long range of the Coulomb potential $\sim 1/r$. In fact, if one naïvely tries to treat a plasma as a gas mixture, but with the Coulomb inter-particle potential, then the integrals of the collision operator diverge [61]. One of the defining characteristics of plasmas is their ability to support collective effects. The classic example is Debye screening, in which every particle in a plasma carries with it a cloud of opposite charge that effectively reduces the range of the charge's potential to the Debye length, λ_D. Interactions

between particles over distances larger than the Debye length are mediated by the electric and magnetic fields, whilst only collisions at distances shorter than the Debye length are considered to be true collisions. Furthermore, the effect of small angle collisions vastly outweighs the effect of the occasional large angle deflection in Coulomb systems [10]. The kinetic equation obtained by cutting of the range off the potential at the Debye length and expanding the Boltzmann equation around grazing collisions is the Landau equation [55],

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{x}} + \vec{F}\cdot\frac{\partial}{\partial \vec{v}}\right) f^{\alpha}(\vec{x},\vec{v},t) = \sum_{\beta} C_{\alpha\beta}(f^{\alpha}, f^{\beta}), \tag{4.32}$$

where

$$C_{\alpha\beta}(f^{\alpha}, f^{\beta}) = 2\pi\frac{e_{\alpha}^2 e_{\beta}^2 \ln\Lambda}{m_{\alpha}}\frac{\partial}{\partial \vec{v}}\cdot\int d\vec{v}'\left(\frac{\hat{I}}{g} - \frac{\vec{g}\vec{g}}{g^3}\right)\cdot\left(\frac{1}{m_{\alpha}}\frac{\partial}{\partial \vec{v}} - \frac{1}{m_{\beta}}\frac{\partial}{\partial \vec{v}'}\right) f^{\alpha}(\vec{v}) f^{\beta}(\vec{v}'). \tag{4.33}$$

Here $\vec{F}$ is the external force that act on the plasma, plus the self-consistent fields that are governed by the Maxwell equations. As we have seen, a more satisfactory derivation of the Landau equation may be obtained from the BBGKY hierarchy by an expansion in the plasma parameter.

The properties we have discussed here carry over to the Landau equation, and the Chapman-Enskog method applied with only minor modifications.

4.2 Chapman-Enskog method applied to a plasma

The application of the Chapman-Enskog method to a two component (electron-ion) plasma described by the Landau equation (4.32) and the careful calculation of all the transport coefficients was first accomplished by Braginskii. His monograph has since become a standard reference [1]. One of the characteristics of the plasma case is the disparity of mass between the electron and ion components. This large mass difference

($\sim 1:1836$ for a hydrogen plasma) impairs the exchange of energy between the two species and as a consequence, the approach of the two species to a thermodynamic equilibrium with one common temperature (4.5) guaranteed by the H-Theorem is slow. Much faster is the equilibration of the particles of the same species amongst themselves to a common temperature T_α as their collisions are much more efficient at exchanging energy. In the plasma, the electron component is the first to reach equilibrium in a time of the order of the electron collision time

$$\tau_e = \frac{3\sqrt{m_e}T_e^{3/2}}{4\sqrt{2\pi}Ze^4 n_e \Lambda_e}, \tag{4.34}$$

followed by the ion component in a time

$$\tau_i = \frac{3\sqrt{m_i}T_i^{3/2}}{4\sqrt{\pi}Z^4 e^4 n_i \Lambda_i}, \tag{4.35}$$

that is longer by the square root of the mass ratio $\sqrt{m_i/m_e}$ for a plasma with ion charge $Z = 1$, and equal temperatures, $T_i = T_e$. Only on a time scale a further $\sqrt{m_i/m_e}$ longer than this is an absolute thermal equilibrium with one single temperature achieved (4.5). In Braginskii's method the square root of the mass ratio $\sqrt{m_e/m_i}$ is treated as being of the same order as the Chapman-Enskog expansion parameter δ, that is the ratio of the collision time to the hydrodynamic time (4.14). The collision operators can be expanded in the mass ratio $\sqrt{m_e/m_i}$ with the result that the solution to the ion and electron kinetic equations can proceed almost *independently*. Braginskii's treatment also allows for the presence of an external magnetic field, but this is ignored for the sake of clarity in the sketch presented below.

In common with the Hilbert expansion, the distribution function for each component, $\alpha = e, i$ is expanded

$$f^\alpha(\vec{x}, \vec{v}, t) = \sum_{n=0}^{\infty} \delta^n f^{(n)}, \tag{4.36}$$

and the kinetic equation is written in the form (4.15),

$$C_{ee}(f^e, f^e) + C'_{ei}(f^e, f^{i'}) =$$

$$\frac{df^e}{dt} + \vec{v}\frac{\partial f^e}{\partial \vec{x}} - \left(\frac{e}{m_e}\vec{E} + \frac{d\vec{u}_e}{dt}\right)\frac{\partial f_e}{\partial \vec{v}} - \frac{\partial \vec{u}_e}{\partial \vec{x}} : \vec{v}\frac{\partial f^e}{\partial \vec{v}}$$
$$-C'_{ei}(f_e, f_i - f'_i) - C''_{ei}(f_e, f_i). \tag{4.37}$$

but Braginskii takes into account just the first term in the smallness parameter δ

$$f_a = f_a^0(1 + \Phi). \tag{4.38}$$

This form is substituted into the kinetic equation and the zeroth order equations are obtained by setting the collision terms, which have been simplified to lowest order in the mass ratio, equal to zero

$$C_{ee}(f_e^0, f_e^0) + C'_{ei}(f_e^0) = 0, \tag{4.39}$$

$$C_{ii}(f_i^0, f_i^0) = 0. \tag{4.40}$$

These have separate local Maxwellians as solutions

$$f_a^0(\vec{x}, t) = n_a(\vec{x}, t)\left(\frac{m_a}{2\pi T_a(\vec{x}, t)}\right)^{3/2} \exp -\frac{m_a}{2T_a(\vec{x}, t)}(\vec{v} - \vec{U}_a(\vec{x}, t))^2, \tag{4.41}$$

in accordance with our discussions. Recall that unlike the Hilbert expansion [63], f_a^0 determines the full hydrodynamic moments in the Chapman-Enskog method. The density n^a, velocity $\vec{u}^a$ and temperature T^a are determined by the equations (4.1-4.3) on the replacement of f^a with f_a^0. This places conditions on the correction term,

$$\int d\vec{v}\, f^0\Phi = 0, \quad \int d\vec{v}\, \vec{v} f^0\Phi = 0, \quad \int d\vec{v}\, v^2 f^0\Phi = 0. \tag{4.42}$$

To the first order in the Chapman-Enskog parameter, in electron equation (4.37), the collision term is linearized in Φ and f_a^0 only is substituted into the small terms on the right hand side. Most important, is the treatment of the time derivative of f_a^0 which is expressed by the conserved moments, i.e. it has a normal form (4.24). As we have already discussed, the result of this approximation is an inhomogeneous integral equation for the correction Φ. The solubility conditions of the inhomogeneous

integral equation are such that the right hand side of (4.37) must be orthogonal to the solutions of the homogeneous equation, $\Phi^h = \{1, \vec{v}, v^2\}$. This condition forces the conserved moments (4.1-4.3) to satisfy the Euler equations, which are the fluid moment equations neglecting dissipation. These in turn can be used to eliminate the time derivatives of the conserved moments that arise from the normal form of f_α^0 (4.29). The result is the following linear, inhomogeneous integral equation for the correction term,

$$C_{ee}(f_e^0, f_e^0\Phi) + C_{ee}(f_e^0\Phi, f_e^0) + C_{ei}(f_e^0\Phi) = f_e^0\left[\left(\frac{v^2}{2v_{Te}^2} - \frac{5}{2}\right)\vec{v}\cdot\nabla \ln T_e + \left(3\sqrt{\frac{\pi}{2}}\frac{v_{Te}^3}{v^3} - 1\right)\frac{1}{\tau_e v_{Te}^2}\vec{U}\cdot\vec{v} + \frac{1}{n_e T_e}\vec{R}^1\cdot\vec{v} + \frac{1}{2v_{Te}^2}\left(v_i v_j - \frac{v^2}{3}\delta_{ij}\right)W_{ij}\right]. \quad (4.43)$$

Notice that the inhomogeneous term on the right hand side of (4.43) is proportional to the effects that disturb the equilibrium, i.e. gradients in the temperature ∇T_e, strains W_{ij}, flow velocity $\vec{U}$, and the friction $\vec{R}$. Because the equation (4.43) is linear, the solution will also be proportional to these terms. Evaluating the nonhydrodynamical moments corresponding to heat flux $\vec{q}_e$, the traceless part of the pressure tensor σ_{ij}, etc. that are determined by the correction Φ will give the heat flux due to each perturbing factor on the right hand side of (4.43) and so on. In this way closure to the moment equations is achieved and the transport coefficients defined. There are some extra details that have been omitted in the derivation of (4.43) that result from the exploitation of the smallness of the mass ratio. The electron ion collision term to lowest order in the mass ratio is independent of the ion distribution function, and it is this form that appears on the right hand side, $C_{ei}(f_e^0\Phi)$. The correction terms, from the full e-i collision term are grouped with the small terms on the right hand side and are responsible for the terms proportional to $\vec{U}$ and the correction to the friction term $\vec{R}^1$.

The ion kinetic equation is linearized in a similar way, but as the ion distribution function is only weakly effected by collisions with electrons, the whole of the cross collision term C_{ie} is grouped with the small terms on the left hand side. Due to this, the ion equation has the same form as for a single component gas, and the ion distribution function is determined entirely by ion-ion collisions.

$$C_{ii}(f_i^0, f_i^0\Phi) + C_{ii}(f_i^0\Phi, f_i^0) = f_i^0\left[\left(\frac{v^2}{2v_{Ti}^2} - \frac{5}{2}\right)\vec{v}\cdot\nabla \ln T_i + \frac{1}{2v_{Ti}^2}\left(v_i v_j - \frac{v^2}{3}\delta_{ij}\right)W_{ij}\right]. \tag{4.44}$$

The technical problem now centers around the solution to the integral equations (4.43) and (4.44). As in the original Chapman-Enskog method, Braginskii employed an expansion in Sonine-Laguerre polynomials, that reduces the integral equation into a system of algebraic equations.

4.2.1 Solution to the first order equations

To complete the solution, tensor invariance of (4.43) and (4.44) leads to the following form of the correction

$$\Phi(\vec{v}) = \Phi_i(v^2)v_i + \Phi_{ij}(v^2)\left(v_i v_j - \frac{v^2}{3}\delta_{ij}\right). \tag{4.45}$$

The vector term Φ_i corresponds to the vector perturbations of temperature gradients ∇T and flow macroscopic flow velocity $\vec{U}$, whilst the tensor term Φ_{ij} corresponds to the rate of strain W_{ij}. The heat flux $\vec{q}$ and momentum transfer due to collisions $\vec{R}^1$ will be determined by the vector part only, and the stress tensor σ_{ij} is determined by the tensor part, Φ_{ij}.

In order to give a concrete example, consider the source terms that are proportional to ∇T_e. By inspection the solution for the vector part Φ_i must be of the following form

$$\Phi_i(v^2) = A(v^2)\nabla \ln T_e, \tag{4.46}$$

where A is a scalar function. On adopting this form (4.46) for Φ_i, it is found that the function A must satisfy the single integral equation,

$$C_{ee}(f_e^0, f_e^0 A\vec{v}) + C_{ee}(f_e^0 A\vec{v}, f_e^0) + C_{ei}(f_e^0 A\vec{v}) = f_e^0 \left(\frac{v^2}{2v_{Te}^2} - \frac{5}{2}\right)\vec{v}. \tag{4.47}$$

The quantity $A(v^2)$ is then expanded in Sonine-Laguerre Polynomials,

$$A(v^2) = \tau_e \sum_{k=1}^{\infty} a_k L_k^{(3/2)}(x), \quad x = \frac{v^2}{2v_{Te}^2}, \tag{4.48}$$

which replaces the integral equation by the infinite set of algebraic equations

$$\sum_{l=1}^{\infty} A_{kl} a_l = \delta_{1k}, \quad k = 1, 2, \ldots \tag{4.49}$$

and where A_{kl} is a dimensionless matrix

$$\begin{aligned} A_{kl} = -\frac{4\tau_e}{15 n_e} \frac{1}{2v_{Te}} \Big[& \int d\vec{v}\, L_k^{(3/2)}(x) v_i C_{ee}(f_e^0, f_e^0 L_l^{(3/2)}(x) v_i) \\ & + \int d\vec{v}\, L_k^{(3/2)}(x) v_i C_{ee}(f_e^0 L_l^{(3/2)}(x) v_i, f_e^0) \\ & + \int d\vec{v}\, L_k^{(3/2)}(x) v_i C_{ei}(f_e^0 L_l^{(3/2)}(x) v_i) \Big], \end{aligned} \tag{4.50}$$

containing matrix elements of both the electron-electron and electron-ion collision operators. A solution may be obtained by truncating the infinite set of equations (4.49) at some level. From this solution the terms in the heat flux and friction arising from temperature gradients can now be written in terms of the expansion coefficients a_l,

$$\vec{q}_T = -\frac{5}{2} \frac{n_e T_e \tau_e}{m_e} a_1 \nabla T_e, \tag{4.51}$$

$$\vec{R}_T = -\frac{5}{2} n_e \sum_{k=1}^{\infty} A_{0k} a_k \nabla T_e. \tag{4.52}$$

The contributions due to the relative velocity $\vec{U}$ are found in a similar manner. With the explicit definition of the transport coefficients, we have reduced the kinetic equation to real plasma hydrodynamics. The validity conditions for the procedure are

again the smallness of the Chapman-Enskog parameter, which is ensured by separation of collisional and hydrodynamic scales (4.12) and (4.13).

The Sonine-Laguerre polynomial expansion (4.48) converges rapidly, and working with the two polynomial approximation Braginskii has obtained the following results for the closure relations. The heat fluxes are found to be

$$\vec{q}_e = \vec{q}_u + \vec{q}_T = 0.71\, n_e T_e \vec{u} - \kappa_e \nabla T_e, \tag{4.53}$$

$$\vec{q}_i = -\kappa_i \nabla T_i \tag{4.54}$$

where

$$\kappa_e = 3.16\, \frac{n_e T_e \tau_e}{m_e}, \quad \text{and} \quad \kappa_i = 3.9\, \frac{n_i T_i \tau_i}{m_i}. \tag{4.55}$$

All the electron coefficients contain contributions from both electron-electron and electron-ion collisions (4.50). The relative importance of these two effects is dependent upon the ionization Z, with the electron-ion collisions occuring Z times more frequently. Because of this the electron transport coefficients are Z dependent, and have been tabulated by Braginskii for $Z = 1, 2, 3, 4, \infty$. The electron thermal conductivity coefficient (4.55) $3.16 = \zeta(1)$ can be given for arbitrary charge $\zeta(Z)$ by the simple formula

$$\zeta(Z) = 4.42\, \frac{Z + 1.7\, Z^2}{0.8 + 2.4\, Z + 0.6\, Z^2}. \tag{4.56}$$

The pressure tensor for each species is determined by

$$\sigma_{ij} = -\eta\, W_{ij} \tag{4.57}$$

where the electron and ion viscosities are determined to be

$$\eta_e = 0.73\, n_e T_e \tau_e, \quad \text{and} \quad \eta_i = 0.96\, n_i T_i \tau_i. \tag{4.58}$$

The friction and heat generation are

$$\vec{R} = -\frac{m_e n_e}{\tau_e}\, 0.51\, \vec{u} - 0.71\, n_e \nabla T_e, \tag{4.59}$$

and the heat generation in the ion component

$$Q_i = \frac{3m_e}{m_i}\frac{n_e}{\tau_e}(T_i - T_e), \tag{4.60}$$

and in the electron component

$$Q_e = -\vec{R}\cdot\vec{u} + \frac{3m_e}{m_i}\frac{n_e}{\tau_e}(T_i - T_e). \tag{4.61}$$

One of the key assumptions of the Chapman-Enskog method was the separation of scales between collisions and the hydrodynamic variables. We will now examine the limits of validity of (4.12) and (4.13).

4.3 Nonlocal transport

The conditions of applicability of the Chapman-Enskog method (4.12, 4.13) are easy to realize in a neutral gas, but because of the wide range of characteristic scales exhibited by plasmas, it is not hard to find conditions where the opposite is true

$$L_H \ll l_{\alpha\alpha'}, \tag{4.62}$$

$$\tau_H \ll \tau_\alpha. \tag{4.63}$$

In the neutral gas case, this limit is very uninteresting and merely describes the free streaming of the gas particles. In the plasma case conditions (4.62, 4.63) correspond to the complicated interactions via the self-consistent fields that are described by the Vlasov equation (3.36). Vlasov theory is usually more complicated than hydrodynamics because of the need to describe the full dependence in velocity space of the one particle distribution function $f^\alpha(\vec{x},\vec{v},t)$. In contrast the fluid theory is only concerned with the evolution of the hydrodynamic variables $n_\alpha(\vec{x},t)$, $\vec{u}_\alpha(\vec{x},t)$, and $T_\alpha(\vec{x},t)$. Because of this, velocity space effects such as Landau damping are usually deemed inaccessible by fluid theories.

Fluid theory and Vlasov theory are two limiting cases of the general description in terms of the Landau equation (4.32). In the first case, the collision term is dominant, and in the other it is negligible. Numerical solutions of the Fokker-Planck equation have shown the inequality (4.12) for the applicability of Chapman-Enskog theory in describing thermal transport takes the form

$$L_H \lesssim 0.01\, l_{ei}. \tag{4.64}$$

Deviation from classical heat flow is found to occur when the gradient scale length is still on the order of a hundred mean free paths [3]. This somewhat surprising result is understandable, because in the case of (4.64) the heat flux is carried by the fast electrons in the tail of the distribution function which have mean free paths much longer than the thermal average [52]. Condition (4.64) is of practical relevance to the modeling of Inertial Confinement Fusion (ICF) plasmas and has lead to various proposals for modifications to the classical heat flow formula (4.53, 4.55) ($u_i = 0$),

$$\vec{q}_e^{Brag}(\vec{x},t) = -\kappa_e^{Brag}\nabla T_e(\vec{x},t). \tag{4.65}$$

Note that classical transport (4.65) is local in both space and time, but any theory of plasma transport not restricted by the conditions (4.12, 4.13) will necessarily be nonlocal due to the mixing of hydrodynamic and collisional scales. Luciani *et al.* [50] have proposed delocalizing the expression (4.65), by convolving it with some kernel $w(x,x')$

$$q_e(x) = \int dx'\; q_e^{Brag}(x')w(x,x'). \tag{4.66}$$

The form of the kernel was phenomenological, but has been given analytic justification [56]. Albritton *et al.* [51] have proposed a similar expression to (4.66) by solving a simplified version of the original Fokker-Planck equation describing the fast unthermalized electrons responsible for the heat flow. In Ref. [52] Epperlein and Short have compared the nonlocal expressions [56, 51], by numerically calculating the decay rate

of a thermal fluctuation in one dimension

$$T_e(x, t = 0) = T_0 + \delta T_e(0) \exp(ikx) \tag{4.67}$$

described by

$$\frac{3}{2} n_e \frac{\partial T_e}{\partial t} + \nabla \cdot \vec{q}_e = 0. \tag{4.68}$$

with the heat flux given by the nonlocal expressions (4.66), from Refs. [50, 51] and that of the Fokker-Planck simulation. Neither of the models [50, 51] agreed particularly well with the Fokker-Planck results. Because of this Epperlein and Short have proposed their own nonlocal formula which reproduces the correct wavenumber dependent decay rate for the fluctuation (4.67) in comparison with their Fokker-Planck simulations [52]. They also raise the issue that the correct form of the nonlocal expression may be dependent upon the type of perturbation considered. For example, the relaxation of temperature in a thermal wave (4.67) could differ from that in an ion acoustic wave, and also be affected by the presence of a laser pump, which modifies the form of the distribution function due to inverse bremsstrahlung heating.

The nonlocal formulas [50, 51, 52] are intended to describe transport in the weakly collisional regime roughly defined by the inequality [54]

$$0 < k l_{ei} \lesssim \frac{1}{\sqrt{Z}}. \tag{4.69}$$

In the context of magnetic fusion research Hammett and Perkins [58] have shown how plasma response in the collisionless regime (4.62), may also be reproduced in fluid moment equations, by the appropriate choice of closure. This work was motivated by the relative simplicity of the fluid models over Vlasov models, being both easier to solve and more transparent to physical insight. The fluid moment equations lead to rational polynomial expressions for the electron and ion susceptibilities. The transport coefficients are then chosen so as to reproduce the correct collisionless forms for the susceptibilities.

A combination of the ideas of delocalizing thermal transport, and the inclusion of collisionless response into fluid moment equations [58] has led Berger *et al.* to include a novel closure into the fluid equations in their 3D fluid code, for the purpose of investigating the filamentation of laser light in ICF plasmas [65]. The importance of the proper treatment of nonlocal effects for the understanding of filamentation has also been discussed by Epperlein *et al.* [53] in connection with the experiments of Young [66]. This nonlocal closure used in the investigations of Ref. [65] is explained by Kaiser *et al.* [67] and is an attempt to provide a closure valid for arbitrary ratio of perturbation wavenumber k to electron-ion and ion-ion mean free paths, kl_{ei} and kl_i. The method is similar to Hammett- Perkins, but the transport coefficients are allowed to possess arbitrary wavenumber dependence, determined by requiring the plasma response to agree with that from a kinetic Fokker-Planck analysis. The ion transport coefficients are chosen in such a way as to reproduce the correct ion damping compared to the Fokker-Planck results of [71, 72] and the electron thermal conductivity is taken to be a smooth interpolation between the nonlocal models of [51] and the Hammett-Perkins value [58], which produces the correct collisionless value for electron Landau damping for $kl_{ei} \gg 1$.

We will now describe a theory for closure to the hydrodynamic moment equations, also capable of describing transport occurring with an arbitrary ratio of scales, including the collisionless Vlasov regime (4.62, 4.63). In contrast to the method of Kaiser *et al.* [67] this is not achieved by a fitting procedure, but rather from the solution to the kinetic equation (4.32) for electrons and ions on the assumption that the plasma is close to equilibrium, highly ionized $Z \gg 1$, the temperature ratio ZT_e/T_i large and the plasma motions quasineutral. Nonlocal electron transport is described in section 4.4, and has some similarities with the Chapman-Enskog method. The result is a full set of nonlocal (ω and k dependent) transport coefficients. Ion transport is approached via a different method (Grad 21M) and is described in Section 4.5.

The ion closure is in terms of temporally nonlocal viscosity and thermal conduction. The resulting closures reproduce the correct dispersion and damping of ion acoustic waves over the whole range of electron and ion particle collisionality when compared to Fokker-Planck simulations [75, 71, 72, 73].

4.4 Nonlocal electron transport

4.4.1 Legendre decomposition of the kinetic equation

The electron kinetic equation is linearized with respect to a small perturbation, which may be interpreted as either the correction to the single particle distribution function, or with the results of the last chapter the phase space fluctuation,

$$f^e(\vec{x},\vec{v},t) = F_0^e + \delta f_e(\vec{x},\vec{v},t). \tag{4.70}$$

We restrict ourselves to the isotropic case and expand this phase space fluctuation in Legendre polynomials,

$$\delta f_e(v,\mu) = \sum_{l=0}^{\infty} P_l(\mu)\delta f_l(v), \tag{4.71}$$

where $\mu = \vec{k}\cdot\vec{v}/kv$. From the electron kinetic equation we obtain an infinite set of equations for the angular harmonics,

$$\frac{\partial \delta f_0}{\partial t} + \frac{i}{3}kv\delta f_1 - \frac{i}{3}kvu_i\frac{\partial F_0}{\partial v} = C_{ee}[\delta f_0], \qquad (l=0), \tag{4.72}$$

$$ikv\delta f_0 + \frac{2}{5}ikv\delta f_2 + i\frac{e}{m_e}k\delta\phi\frac{\partial F_0}{\partial v} = -\nu_{ei}\delta f_1, \qquad (l=1), \tag{4.73}$$

$$\frac{2}{3}ikv\delta f_1 + \frac{3}{7}ikv\delta f_3 - \frac{2}{3}ikvu_i\frac{\partial F_0}{\partial v} = -3\nu_{ei}\delta f_2, \quad (l=2), \tag{4.74}$$

$$\frac{l}{2l-1}ikv\delta f_{l-1} + \frac{l+1}{2l+3}ikv\delta f_{l+1} = -\frac{1}{2}l\,(l+1)\nu_{ei}\delta f_l, \qquad (l>2). \tag{4.75}$$

It is necessary to examine the different terms in this set of equations and justify the approximations that have been made. The terms arising from the time derivative

in the kinetic equation are small as we are interested in low frequency phenomena $\omega \sim kc_s \ll kv_{Te}$ where $c_s = \sqrt{ZT_e/m_i}$. Because of this, the time derivative has been kept only to the lowest order in the equation for the symmetric part of the distribution function (4.72). Formally the stationary approximation for angular harmonics with $l \geq 1$ assumes that $\omega \ll \nu_{ei}$ but in fact it is also valid in the opposite limit as far as the wave phase velocity is small, $\omega/k \ll v_{Te}$. The collision operators are defined in the ion reference frame, and therefore the ion velocity shows up in the $l = 0$ (4.72) and $l = 2$ (4.74) equations. These correspond to compression and viscosity respectively. In the $l = 1$ equation (4.73) the term proportional to $\delta\phi$ is the acceleration in the perturbed potential. The advection term in the kinetic equation $\vec{v} \cdot \partial f/\partial \vec{v}$ couples the l equation to the $l+1$ and $l-1$ equations as a result of the identity for Legendre polynomials

$$\mu P_l(\mu) = \frac{l+1}{2l+1} P_{l+1}(\mu) + \frac{l}{2l+1} P_{l-1}(\mu). \tag{4.76}$$

In fact this advection term is solely responsible for the coupling of the equations for the harmonics with $l > 2$ (4.75). In the equations for all harmonics with $l > 0$, the term on the right hand side is the angular scattering of electrons due to collision with ions. The term is especially simple as Legendre polynomials are eigenfunctions of the electron-ion collision operator in the Lorentz approximation,

$$C_{ei}[P_l(\mu)\delta f_l(v)] = -\frac{1}{2} l(l+1)\nu_{ei}\delta f_l(v). \tag{4.77}$$

In the symmetric, $l = 0$ equation

$$C_{ee}[\delta f_0] \equiv C_{ee}[F_0, \delta f_0] + C_{ee}[\delta f_0, F_0] \tag{4.78}$$

is the linearized isotropic part of the electron-electron collisional operator. The expression for the electron-electron collision term C_{ee} is more complicated because it is nonlinear with respect to f_e and contains integral terms [60]. In highly ionized plasmas, $Z \gg 1$, the collision term C_{ee} is Z times smaller than the electron-ion collision term but is still important because it is responsible for energy redistribution

between the electrons. Epperlein has shown this in his numerical investigation of the ion acoustic damping rate [75]. For this reason we only account for electron-electron collisions in the equation for the symmetric part of the electron distribution function (4.72) because here the electron-ion collision term does not contribute.

Landau damping and a renormalized collision frequency

The usual strategy for solving this infinite hierarchy of coupled equations rests on the assumption that the higher harmonics are small. In the collisional region a good approximation may be achieved by truncating the series at some $l = l_{max}$ and setting $\delta f_l = 0$, for $l > l_{max}$ (usually $l_{max} \sim 2$) [49, 59]. The approximation is very good for the strongly collisional region and can be expanded into the more weakly collisional region (by the inclusion of more polynomials), but such a scheme can never capture collisionless Landau damping. To see why this is so, let us briefly consider the source of Landau damping, or as it is sometimes called "phase mixing". Landau damping results from the advective term in the kinetic equation, as can be appreciated from the following example, due to Hammett and Perkins, [58]. Let the distribution function evolve from an initial perturbation in density

$$\delta f_e(x, v, t = 0) = \delta n(0) \exp(ikx) F_0(v) \tag{4.79}$$

according to the equation

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) \delta f_e(x, v, t) = 0, \tag{4.80}$$

i.e. simple advection in 1D. At time t the perturbation in the distribution function will be given by

$$\delta f_e(x, v, t) = \delta n(0) \exp ik(x - vt) F_0(v), \tag{4.81}$$

and the initial perturbation in the density will be damped away due to phase mixing,

$$\delta n(t) = \delta n(0)\, exp(ikx) \int_{-\infty}^{\infty} dv\, \exp(-ikvt) F_0(v) \tag{4.82}$$

$$= \delta n(0) \exp -\frac{k^2 v_{Te}^2}{2} t^2. \tag{4.83}$$

Alternatively, Landau damping may also be viewed as a resonance effect. For example, on Laplace transforming (4.80) in time with the initial condition (4.79)

$$\delta n(x,p) = i\delta n(0) \exp(ikx) \int dv \frac{F_0}{p - kv}, \tag{4.84}$$

and since the electron thermal velocity greatly exceeds the plasma velocity $v_{Te} \gg \omega/k$, the resonance, and hence damping comes from the electrons that propagate almost across the wavevector. These electrons are described by high l harmonics.

A summation procedure that avoids truncation and includes all the angular harmonics has been described by [56, 57]. The idea consists of the solving of Eq. (4.75). Putting the second term on the left hand side of this equation into its right hand side and introduce the modified collision frequency

$$\tilde{\nu}_l = \frac{1}{2} l\,(l+1)\nu_{ei} + ikv \frac{l+1}{2l+3} \frac{\delta f_{l+1}}{\delta f_l}, \tag{4.85}$$

then the formal solution of Eq. (4.75) reads

$$\delta f_l = -i \frac{l}{2l-1} \frac{kv}{\tilde{\nu}_l} \delta f_{l-1}. \tag{4.86}$$

If we substitute this solution (4.86) back into (4.85), then a recurrent formula for $\tilde{\nu}_l$ appears

$$\tilde{\nu}_{l-1} = \frac{1}{2} l\,(l-1)\nu_{ei} + \frac{l^2}{4l^2-1} \frac{k^2 v^2}{\tilde{\nu}_l} \tag{4.87}$$

which completes the formal solution of Eq. (4.75). In fact, it is enough to calculate $\tilde{\nu}_1 = \nu_{ei} H_1(kv/\nu_{ei})$, because all necessary functions can be expressed through it explicitly. The function H_1 can be written as a continued fraction, but in Ref. [57] the simple approximation was proposed $H_1(x) = [1 + (\pi x/6)^2]^{1/2}$ which has the proper asymptotics and deviates from exact solution by less than 10% when $x \sim 1$. The set of equations for the angular harmonics (4.72-4.74) is now closed, and can be written for δf_0 by back substitution.

4.4.2 Nonlocal hydrodynamic closure

So far we have performed a Legendre decomposition of the kinetic equation and made approximations that have left us with the much simplified equation for the symmetric part,

$$\frac{\partial \delta f_0}{\partial t} - \frac{i}{3} k v \delta f_1 + \frac{i}{3} k v \delta u_i \frac{\partial F_0}{\partial \vec{v}} = C_{ee}[\delta f_0], \tag{4.88}$$

$$ikv\delta f_0 + i\frac{e}{m_e} k \delta\phi \frac{\partial F_0}{\partial \vec{v}} - (\tilde{\nu}_1 - \nu_{ei}) \delta u_i \frac{\partial F_0}{\partial \vec{v}} = -\tilde{\nu}_1 \delta f_1. \tag{4.89}$$

The important points are: The effects of all higher order harmonics and hence Landau damping are contained within the renormalized collision frequency, $\tilde{\nu}_1$, and electron-electron collisions are only accounted for in the equation for the symmetric part, which is consistent with the assumption of large Z. The next step is the reduction of this kinetic equation into a closed set of fluid equations. This begins by solving (4.88) and (4.89) for δf_0 and taking the Laplace transform in time, which introduces the initial condition,

$$\left(\frac{k^2 v^2}{3\nu_1} + p\right)\left(\delta f_0 - \frac{e\delta\phi}{T_0} F_0\right) - C_{ee}[\delta f_0] = -p\frac{e\delta\phi}{T_e} F_0 - iku_i \frac{v^2}{3v_{Te}^2} \frac{\nu_{ei}}{\nu_1} F_0 + \delta f_0(v, t=0). \tag{4.90}$$

Central to the Chapman-Enskog method was the assumption of the normal form of the distribution function (4.24). Such a form is not assumed here *except* at the initial moment in time. We can consider that the system was prepared in such a way as the only fluctuations correspond to those of temperature and density [8],

$$\delta f_0(v, 0) = \left[\frac{\delta n(0)}{n_e} + \frac{\delta T(0)}{T_e}\left(\frac{v^2}{2v_{Te}^2} - \frac{3}{2}\right)\right] F_0(v) \tag{4.91}$$

With this choice of initial condition and due to the linear form of the equation (4.90) for δf_0 the solution can be expressed as the linear combination of the three basis functions ψ^A $(A = N, R, T)$,

$$\delta f_0 = \frac{e\delta\phi}{T_0} F_0 + \sum_A P_A \psi^A F_0. \tag{4.92}$$

We have made the definitions, $P_N = \delta n_e(0)/n_e - pe\delta\phi/T_e$, $P_T = 3\delta T_e(0)/2T_e$, and $P_R = -ik\delta u_i$. The ψ^A in (4.92) must then satisfy the equations

$$\left(\frac{k^2v^2}{3\nu_1} + p\right)\psi^A = F_0^{-1}C_{ee}[F_0\psi^A] + S_A, \tag{4.93}$$

with the three different source terms $S_N = 1$, $S_T = v^2/3v_{Te}^2 - 1$, and $S_R = v^2\nu_{ei}/3v_{Te}^2\nu_1$. This is analogous to the integral equation (4.43) in the Chapman-Enskog method, where the right hand side of the equation for the correction to the distribution function is given by factors that disturb the equilibrium, i.e. gradients in temperature etc. Here the source terms correspond to the disturbance in electric potential, ion velocity and the initial perturbations in density and temperature. If one now calculates the present density $\delta n_e(t)$ and temperature $\delta T_e(t)$ from the solution for δf_0 (4.92),

$$\delta n = 4\pi\int_0^\infty dv\, v^2\delta f_0, \tag{4.94}$$

$$\delta T = \frac{4\pi m_e}{2n_e}\int_0^\infty dv\, v^2(v^2 - 3v_{Te}^2)\delta f_0, \tag{4.95}$$

then the initial conditions P_N and P_T may be eliminated. Putting this solution into the equation for δf_1 (4.89) gives,

$$\delta f_1 = -\frac{eE^*v}{T_0\nu_1}\frac{J_T^T\psi^N - J_T^N\psi^T}{D_{NT}^{NT}}F_0 - i\frac{kv}{\nu_1}\frac{\delta T}{T_0}\frac{(J_N^N + J_N^T)\psi^T - (J_T^T + J_T^N)\psi^N}{D_{NT}^{NT}}F_0 - \frac{v\delta u_i}{\nu_1 v_{Te}^2}\left[\nu_1 - \nu_{ei} + k^2v_{Te}^2\left(\psi^R + \frac{D_{NT}^{TR}}{D_{NT}^{NT}}\psi^N - \frac{D_{NT}^{NR}}{D_{NT}^{NT}}\psi^T\right)\right]F_0, \tag{4.96}$$

where the following S_B moments of the ψ^A have been introduced,

$$J_B^A = \frac{4\pi}{n_{0e}}\int_0^\infty v^2dv\psi^A F_0 S_B, \quad \text{and} \quad D_{AB}^{CD} = J_A^C J_B^D - J_A^D J_B^C. \tag{4.97}$$

With δf_1 given by (4.96) closure has been achieved. The reason why this expression achieves the closure is because δf_1 is responsible for transport, i.e. current, heat flux and friction. Furthermore, δf_1 is now dependent only upon the conserved moments and as a result, so are the fluxes. The current is given by,

$$\vec{\delta j} = -e\int d\vec{v}\vec{v}\delta f_e, \tag{4.98}$$

$$= -\frac{4\pi e}{3}\int v^2dv\,\delta f_1, \tag{4.99}$$

the heat flux by

$$\delta\vec{q}_e = \frac{1}{2}\int d\vec{v}\vec{v}(m_e v^2 - 5T_e)\delta f_e, \tag{4.100}$$

$$= \frac{2\pi}{3}\int v^2 dv\,(m_e v^2 - 5T_e)\delta f_1, \tag{4.101}$$

and the friction by

$$\delta R_{ie} = m_e \int d\vec{v}\vec{v}\nu_{ei}\delta f^e, \tag{4.102}$$

$$= \frac{4\pi m_e}{3}\int v^2 dv\,\nu_{ei}\delta f_1, \tag{4.103}$$

which will then define the transport coefficients through

$$\delta\vec{j} = \sigma\delta\vec{E}^* + \alpha i\vec{k}\delta T_e + \beta_j e n_e \delta\vec{u}_i, \tag{4.104}$$

$$\delta\vec{q}_e = -\alpha T_e\delta\vec{E}^* - \chi i\vec{k}\delta T_e - \beta_q n_e T_0^e \delta\vec{u}_i, \tag{4.105}$$

$$\delta\vec{R}_{ie} = -(1-\beta_j)n_e e\delta\vec{E}^* + \beta_q n_e i\vec{k}\delta T_e - \beta_r m_e n_e \delta\vec{u}_i v_{Te}/l_{ei}. \tag{4.106}$$

The results are,

$$\sigma = \frac{e^2 n_e}{k^2 T_e}\left(\frac{J_T^T}{D_{NT}^{NT}} - p\right), \quad \alpha = -\frac{e n_e}{k^2 T_e}\left(\frac{J_T^N + J_T^T}{D_{NT}^{NT}} - p\right), \quad \beta_j = 1 - \frac{D_{NT}^{RT}}{D_{NT}^{NT}}, \tag{4.107}$$

$$\beta_q = \frac{D_{NT}^{RT} + D_{NT}^{RN}}{D_{NT}^{NT}}, \quad \chi = \frac{n_e}{k^2}\left(\frac{2J_T^N + J_N^N + J_T^T}{D_{NT}^{NT}} - \frac{5}{2}p\right), \tag{4.108}$$

$$\beta_r = 1 + k^2 v_{Te} l_{ei}\left(J_R^R - (1-\beta_j)(J_R^N + J_R^T) + \beta_q J_R^T\right) - (2\pi)^{3/2}\frac{v_{Te}}{n_e}\int_0^\infty \frac{dv v F_0}{H_1(kv/\nu_{ei})}. \tag{4.109}$$

These algebraic expressions involve the Laplace time variable p. However, in the application of these formulas in describing slow phenomena (on the order of the ion acoustic time-scale) this p dependence is small and can be ignored $p \to 0$. Bychenkov *et al.* [8] have analyzed these expressions, and shown that they possess the correct limit for small wavenumbers, i.e they reduce to the same expressions as Braginskii. For larger wavenumbers, this closure has been shown to produce the correct dispersion and damping of ion acoustic waves over a wide range of wavenumbers, including Landau damping [8].

4.4.3 Laguerre expansion

The three functions ψ^A that solve the integral equation (4.93) and are required for the calculation of the transport coefficients (4.107-4.109) may be obtained via an expansion in terms of generalized Laguerre polynomials $L_n^{(1/2)}(x)$,

$$\psi^A(v) = \left(\frac{Zkl_{ei}}{kv_{Te}}\right) \sum_{n=0}^{\infty} c_n^A L_n^{(1/2)}(v^2/2v_{Te}^2). \tag{4.110}$$

The idea is the same as in the Chapman-Enskog method described in section 4.2.1. The procedure is to expand ψ^A in generalized Laguerre polynomials (4.110), where $x = v^2/2v_{Te}^2$. Next, on multiplying (4.93) by the polynomial $L_m^{(1/2)}(x)$ and weight function $x^{1/2}\exp(-x)$ and integrating over x we convert (4.93) into a linear system of equations for the expansion coefficients, c_n^A.

$$A_{mn}c_n^A = b_m^A \tag{4.111}$$

Here, b_m^A comes from integrating over the source term S_A, and A_{mn} is the sum of two contributions, $A_{mn} = D_{mn} + C_{mn}$ that come from the advective and collision term C_{ee} in (4.93) respectively. The explicit forms are,

$$b_m^A = \int_0^\infty dx x^{1/2} e^{-x} L_m^{(1/2)}(x) S_A \tag{4.112}$$

and

$$\begin{aligned} A_{mn} = {} & \frac{8}{9\sqrt{\pi}} Z(kl_{ei})^2 \int_0^\infty dx \frac{x^3}{H_1} e^{-x} L_m^{(1/2)}(x) L_n^{(1/2)}(x) \\ & + 3\int_0^\infty dx e^{-x} \gamma(3/2, x) L_{m-1}^{(3/2)}(x) L_{n-1}^{(3/2)}(x) \\ & - 2\left[\frac{1}{n-1}\int_0^\infty dx x^{5/2} e^{-2x} L_{n-2}^{(5/2)}(x) L_{m-1}^{(3/2)}(x) + (m \leftrightarrow n)\right]. \end{aligned} \tag{4.113}$$

To see how this arises, note that from [74] the collision term $F_0^{-1}C_{ee}[F_0\psi^A]$ may be written as

$$F_0^{-1}C_{ee}[F_0\psi^A] = F_0^{-1}\frac{2}{Z\sqrt{\pi}}\nu_{ei}(v)v\frac{d}{dv}\left(F_0(v)G\right), \tag{4.114}$$

where G is given by

$$G = \frac{v_{Te}^2}{v}\frac{d\psi^A}{dv}\gamma\left(3/2, v^2/2v_{Te}^2\right) - \frac{2\pi^{3/2}}{3n_0}\left[\int_0^v dw w^3 \frac{d\psi^A}{dw}F_0(w) + v^3\int_v^\infty dw \frac{d\psi^A}{dw}F_0(w)\right]. \tag{4.115}$$

A change of variables to $x = v^2/2v_{Te}^2$ bring this to the form

$$F_0^{-1}C_{ee}[F_0\psi^A] = \frac{kv_{Te}}{Zkl_{ei}}x^{-1/2}e^x\frac{d}{dx}\left(e^{-x}G\right), \tag{4.116}$$

where

$$G = 3\frac{d\psi^A}{dx}\gamma(3/2,x) - 2\left[\int_0^x dy y^{3/2}\frac{d\psi^A}{dy}e^{-y} + x^{3/2}\int_x^\infty dy\frac{d\psi}{dy}e^{-y}\right]. \tag{4.117}$$

On multiplying this through by $x^{1/2}\exp(-x)L_m^{(1/2)}(x)$ and integrating over x, we obtain after one integration by parts,

$$C_{mn} = \int_0^\infty dx \frac{dL_m^{(1/2)}}{dx}e^{-x}\left\{3\frac{dL_n^{(1/2)}}{dx}\gamma(3/2,x) - 2\left[\int_0^x dy y^{3/2}\frac{dL_n^{(1/2)}}{dy}e^{-y} + x^{3/2}\int_x^\infty dy\frac{dL_n^{(1/2)}}{dy}e^{-y}\right]\right\}. \tag{4.118}$$

Now making use of the relations

$$\frac{dL_n^{(1/2)}(x)}{dx} = -L_{n-1}^{(3/2)}(x) \tag{4.119}$$

and

$$\int_0^x dy L_{m-1}^{(3/2)}(y)y^{3/2}e^{-y} = \frac{1}{m-1}L_{m-2}^{(5/2)}(x)x^{5/2}e^{-x} \tag{4.120}$$

and a change in the order of integration in the third term in (4.118), the electron collision part of the result (4.113) follows. A Fortran code has been written by the author that calculates the matrix elements (4.113), solves the matrix (4.111) for the expansion coefficients c_n^A and then determines the transport coefficients (4.107-4.109) for any wavenumber. Details of the workings of this code can be found in Appendix C.1.

4.5 Nonlocal ion transport: Hermitian moment method

Moment methods are an alternative way of solving the kinetic equation. The most famous of these is the Grad 13M method [68]. The distinction between the moment method and the Chapman-Enskog method can be understood as follows. In a moment method the distribution function is expanded in a complete set of orthogonal polynomials. The irreducible tensorial Hermite Polynomials, $H^{(m)}_{r_1\cdots r_\ell}(\vec{c})$ are a good choice because of their orthogonality properties with the Maxwellian weight function [41]. The use of *irreducible* polynomials further clarifies the structure, for example certain moments can be identified with physical quantities. The full (infinite) set of moment equations can be considered, not as a fluid model, but rather as a *bona fide* representation of the kinetic equation. Practically, one must truncate the hierarchy of moment equations at some level, which is where physical approximations must be made. The assumption of the normal form of the distribution function employed in the Chapman-Enskog method is not assumed, and therefore it is sometimes possible to extend the description into the weakly collisional region. We will show that this is indeed the case for high frequency ion transport $\omega \gg \{\nu_i,\ kv_{Ti}\}$ as associated with ion acoustic waves in plasmas with large ZT_e/T_i. This has previously been described in Ref. [64]. Because the usefulness of the (somewhat generalized) Grad 21M method in describing high frequency response is applicable only to the ion component, our discussion will omit the electron part. For a good description of classical transport and the Grad moment method one can refer to the book by Balescu [41].

Following the version of the Grad 21M method described by Balescu, the distribution function is written as

$$f^a(\vec{v},\vec{x},t) = f^{a0}(\vec{v},\vec{x},t)[1+\chi^a(\vec{v},\vec{x},t)] \tag{4.121}$$

where $f^{\alpha 0}$ is a local Maxwellian distribution function (4.5), which determines the hydrodynamic variables n_α, $\vec{u}_\alpha$ and T_α. In the collision dominated regime, the distribution function is close a local Maxwellian, and hence the function χ may be expanded in the Chapman-Enskog smallness parameter δ,

$$\chi = \delta\chi^{(1)} + \delta^2\chi^{(2)} + \dots \tag{4.122}$$

which has the same form as both the Hilbert, and Chapman-Enskog methods. All the information concerning the deviation from this local equilibrium state is contained within the function $\chi^\alpha(\vec{v},\vec{x},t)$. This function is approximated as

$$\chi^\alpha(\vec{x},\vec{c},t) = c_r B_r^\alpha(\vec{x},\vec{c},t) + \left(c_r c_s - \frac{1}{3}c^2\delta_{rs}\right) C_{rs}^\alpha(\vec{c},\vec{x},t) + \cdots \tag{4.123}$$

where $\vec{c} = \sqrt{m_\alpha/T_\alpha}(\vec{v} - \vec{U}^\alpha)$, and the higher order anisotropies have been ignored. Comparing (4.123) with (4.45) of the Chapman-Enskog method, the vector part B_r^α is responsible for heat flux, and is expanded in vector Hermite polynomials, $H_r^{\alpha(2n+1)}(\vec{c})$. The tensor part C_{rs}^α gives the pressure tensor, and is expanded in second order irreducible tensor Hermite polynomials, $H_{rs}^{(\alpha 2n)}(\vec{c})$. The 21M approximation consists of taking

$$c_r B_r^\alpha(\vec{x},\vec{c},t) = h_r^{\alpha(3)}(\vec{x},t)H_r^{(3)}(\vec{c}) + h_r^{\alpha(5)}(\vec{x},t)H_r^{(5)}(\vec{c}) + \dots \tag{4.124}$$

$$\left(c_r c_s - \frac{1}{3}c^2\delta_{rs}\right) C_{rs}^\alpha(\vec{x},\vec{c},t) = h_{rs}^{\alpha(2)}(\vec{x},t)H_{rs}^{(2)}(\vec{c}) + h_{rs}^{\alpha(4)}(\vec{x},t)H_{rs}^{(4)}(\vec{c}) + \dots, \tag{4.125}$$

and ignoring vector polynomials of degree higher than 5, and the second order tensor polynomials of degree higher than 4. The vector Hermite polynomials needed in this approximation are

$$H_r^{(3)}(\vec{c}) = \frac{1}{\sqrt{10}}c_r(c^2 - 5), \tag{4.126}$$

$$H_r^{(5)}(\vec{c}) = \frac{1}{2\sqrt{70}}c_r(c^4 - 14c^2 + 35), \tag{4.127}$$

and the traceless tensors of the second rank are

$$H_{rs}^{(2)}(\vec{c}) = \frac{1}{\sqrt{2}}(c_r c_s - \frac{1}{3}c^2\delta_{rs}), \tag{4.128}$$

$$H^{(4)}_{rs}(\vec{c}) \;=\; \frac{1}{2\sqrt{7}}(c_r c_s - \frac{1}{3}c^2\delta_{rs})(c^2-7). \tag{4.129}$$

Once determined, the unknown space and time dependent coefficients of the expansion (4.124, 4.125) $h^{\alpha(m)}_{r_1\cdots r_q}$ known as the "hermitian moments", will completely define the distribution function (4.121). The $h^{\alpha(m)}_{r_1\cdots r_q}(\vec{x},t)$ are the averages of the Hermite polynomials $H^{\alpha(m)}_{r_1\cdots r_q}(\vec{c})$ over the deviation of the distribution function $\chi^\alpha(\vec{c},\vec{x},t)$ with respect to the Maxwellian weight function,

$$h^{\alpha(m)}_{r_1\cdots r_q}(\vec{x},t) = \int d\vec{c}\,\frac{1}{\sqrt{2\pi}}\exp(-c^2)\; H^{(m)}_{r_1\cdots r_q}(\vec{c})\chi^\alpha(\vec{c},\vec{x},t). \tag{4.130}$$

The first few are identically zero and have not been written in the expansion (4.124, 4.125),

$$h^{\alpha(0)} = 0, \quad h^{\alpha(2)} = 0, \quad \text{and} \quad h^{\alpha(1)}_r = 0. \tag{4.131}$$

This is due to the conditions imposed upon χ^α by the requirement that the Maxwellian part completely determines the conserved moments n_α, T^α and $\vec{u}^\alpha$,

$$\int d\vec{v} f^{\alpha 0}\chi^\alpha = 0, \quad \int d\vec{v} v^2 f^{\alpha 0}\chi^\alpha = 0, \quad \int d\vec{v}\vec{v} f^{\alpha 0}\chi^\alpha = 0. \tag{4.132}$$

Some of these moments are related to quantities of physical importance. Namely, $h^{\alpha(3)}_r$ in (4.124) is related to the heat flux q^α_r, and $h^{\alpha(2)}_{rs}$ in (4.125) the stress tensor σ^α_{rs},

$$q^\alpha_r \;=\; \sqrt{\frac{5}{2}}m_\alpha\Big(\frac{T_\alpha}{m_\alpha}\Big)^{3/2} n_\alpha h^{\alpha(3)}_r \tag{4.133}$$

$$\sigma^\alpha_{rs} \;=\; \sqrt{2}n_\alpha T_\alpha h^{\alpha(2)}_{rs}. \tag{4.134}$$

The moments $h^{\alpha(5)}_r$ and $h^{\alpha(4)}_{rs}$ have no such physical interpretation, however. On numerating the moments, we see there are 5 for n_α, $\vec{u}^\alpha_r$, T_α, 3 components of the heat flux $h^{\alpha(3)}_r$, 5 independent components of the pressure tensor $h^{\alpha(2)}_{rs}$ which amounts to a total of 13 moments. The other 8 come from the nonphysical moments $h^{\alpha(5)}$ (3 moments) and $h^{\alpha(4)}_{rs}$ (5 moments), to a total of 21.

The derivation of the moment equations from the Landau kinetic equation (4.32) is explained in some detail in the book by Balescu [41]. We are not interested in the details of the derivation, and hence we will quote only the important points, and also ignore the magnetic field. The nonlinear equations of evolution of the vector moments are

$$\frac{\partial}{\partial t}h_r^{(1)} = \left(\frac{m_e}{T_e}\right)^{1/2}\left(\frac{e}{m_e}E_r + \frac{1}{m_e n_e}\nabla_r(n_e T_e)\right) + Q_r^{(1)} + U_r^{(1)} + C_r^{(1)} + N_r^{(1)}, \quad (4.135)$$

which is not a true hermitian moment of χ, since this vanishes by (4.132). Rather it is related to the electric current $j_r = en_e\sqrt{T_e/m_e}\,h_r^{(1)}$.

$$\begin{aligned}\frac{\partial}{\partial t}h_r^{a(3)} &= \sqrt{\frac{5}{2}}\left(\frac{T_a}{m_a}\right)^{1/2}\frac{1}{T_a}\nabla_r T_a \\ &\quad + Q_r^{a(3)} + U_r^{a(3)} + D_r^{a(3)} + C_r^{a(3)} + N_r^{a(3)}, \qquad (4.136)\\ \frac{\partial}{\partial t}h_r^{a(5)} &= Q_r^{a(5)} + U_r^{a(5)} + D_r^{a(5)} + C_r^{a(5)} + N_r^{a(5)}. \qquad (4.137)\end{aligned}$$

The second rank tensor moments evolve according to

$$\begin{aligned}\frac{\partial}{\partial t}h_{rs}^{a(2)} &= -\frac{1}{\sqrt{2}}W_{ij} \\ &\quad + Q_{rs}^{a(2)} + U_{rs}^{a(2)} + D_{rs}^{a(2)} + C_{rs}^{a(2)} + N_{rs}^{a(2)}, \qquad (4.138)\\ \frac{\partial}{\partial t}h_{rs}^{a(4)} &= Q_{rs}^{a(4)} + U_{rs}^{a(4)} + D_{rs}^{a(4)} + C_{rs}^{a(4)} + N_{rs}^{a(4)}. \qquad (4.139)\end{aligned}$$

The right hand sides of (4.135, 4.136, 4.138) have a common structure. In the equations for $h_r^{(1)}$ (4.135), $h_r^{a(3)}$ (4.136) and $h_{rs}^{a(2)}$ (4.138), there are terms involving the hydrodynamic variables, the modified electric field $\vec{E} + \nabla(n_e T_e)/en_e$, gradients in temperature ∇T_a and the rate of strains W_{ij}^a. These may be identified with the "thermodynamic forces" that perturb the equilibrium state. There are the collision terms $Q_{\cdots}^{a(n)}$, which arise as the nonhydrodynamical moments are averages of dynamical variables that are not collisional invariants (4.6). These are the generalized frictions,

$$Q_{r_1 r_2}^{a(m)} = n_a^{-1}\int d\vec{v} H_{r_1 r_2 \cdots}^{(m)}\left(\sqrt{\frac{m_a}{T_a}}(\vec{v} - \vec{u}^a)\right) C_{aa'}. \qquad (4.140)$$

In the linear theory, i.e. neglecting all terms of higher than first order in δ (4.122) the only important terms are the source terms $\vec{E} + \nabla(n_e T_e)/en_e$, ∇T_a and W_{ij}^a and the linear parts of the collision terms (4.140). The remaining terms in (4.135-4.139), the "up-term", $U_{\cdots}^{a(n)}$, the "down-term", $D_{\cdots}^{a(n)}$, the "convective terms", $C_{\cdots}^{a(n)}$ and the "non-linear" terms $N_{\cdots}^{a(n)}$ turn out to be negligible and may be discarded [41].

The generalized frictions (4.140) involve matrix elements of the Landau collision operator $C_{aa'}$ (4.33). These may be written in terms of Gaussian integrals and can be evaluated analytically [41]. Our interest is in the ion kinetic equation, and hence we only consider those coming from the linear part of the ion-ion collision operator,

$$C_{ii}(f_i^0, f_i^0 \chi_i^{(1)}) + C_{ii}(f^0 \chi_i^{(1)}, f_i^0). \tag{4.141}$$

The ion-electron collisions have a much smaller effect (by the mass ratio m_e/m_i) as has already been mentioned in the discussion of the Chapman-Enskog method (4.2). The collision terms (4.141) only couple the ion moment equations of the same tensorial nature amongst themselves,

$$\tau_i Q_r^{i(3)} = -c_{33}^i h_r^{i(3)} - c_{35}^i h_r^{i(5)}, \tag{4.142}$$

$$\tau_i Q_r^{i(5)} = -c_{53}^i h_r^{i(3)} - c_{55}^i h_r^{i(5)}, \tag{4.143}$$

$$\tau_i Q_{rs}^{i(2)} = -c_{22}^i h_{rs}^{i(2)} - c_{24}^i h_{rs}^{i(4)}, \tag{4.144}$$

$$\tau_i Q_{rs}^{i(4)} = -c_{42}^i h_{rs}^{i(2)} - c_{44}^i h_{rs}^{i(4)}. \tag{4.145}$$

The coupling coefficients are symmetric under interchange of the indices, due to the symmetry of the collision operator, $c_{ij} = c_{ji}$ evident in (4.33). The vector coupling coefficients are evaluated to be,

$$c_{33} = \frac{2\sqrt{2}}{5}, \quad c_{35} = c_{53} = -\frac{3\sqrt{14}}{35}, \quad c_{55} = \frac{9\sqrt{2}}{14}, \tag{4.146}$$

and the tensor coupling coefficients are

$$c_{22} = \frac{3\sqrt{2}}{5}, \quad c_{24} = c_{42} = -\frac{9\sqrt{7}}{70}, \quad c_{44} = \frac{41\sqrt{2}}{56}. \tag{4.147}$$

We have dropped the i superscript, as the electron coefficients will not be mentioned.

With the expressions (4.142)-(4.145) for the generalized frictions and the explicit values for the coupling coefficients (4.146), (4.147) the the linear theory for the ion vector moments (4.136, 4.137) becomes

$$\frac{\partial}{\partial t}h_r^{i(3)} = -\frac{1}{\tau_i}(c_{33}^i h_r^{i(3)} + c_{35} h_r^{i(5)}) - \sqrt{\frac{5}{2}}\left(\frac{T_i}{m_i}\right)^{1/2}\frac{1}{T_i}\nabla_r T_i, \tag{4.148}$$

$$\frac{\partial}{\partial t}h_r^{i(5)} = -\frac{1}{\tau_i}(c_{53}^i h_r^{i(3)} + c_{55}^i h_r^{i(5)}), \tag{4.149}$$

and the ion tensor moments (4.138), (4.139),

$$\frac{\partial}{\partial t}h_{rs}^{i(2)} = -\frac{1}{\tau_i}(c_{22}^i h_{rs}^{i(2)} + c_{24}^i h_{rs}^{i(4)}) - \frac{1}{\sqrt{2}}W_{ij}, \tag{4.150}$$

$$\frac{\partial}{\partial t}h_{rs}^{i(4)} = -\frac{1}{\tau_i}(c_{42}^i h_{rs}^{i(2)} + c_{44}^i h_{rs}^{i(4)}). \tag{4.151}$$

Equations (4.148-4.151) with the coupling constants (4.146) and (4.147) comprise the description of ion dynamics in the Grad 21M approximation. A solution of these equations will define the heat flux and stress tensor, closing the conservation equations.

4.6 Solution to the Ion moment equations

4.6.1 Classical transport regime

The equations governing the ion dynamics in the 21M approximation (4.148-4.151) are inhomogeneous, first order differential equations. Because of this, we may write their solutions in terms of the Green functions

$$\dot{G}^{(pq)}(t) = -\nu_i \sum_m c_{pm} G^{(mq)}(t), \tag{4.152}$$

$$G^{(pq)}(0) = \delta_{pq}. \tag{4.153}$$

The dot implies a time derivative. The solutions for the moments are formally written as

$$h_r^{i(3)}(t) = \sum_{q=3,5} G^{(3q)}(t) h_r^{i(q)}(0)$$
$$-\nu_i \int_0^t dt' \, G^{(33)}(t') \sqrt{\frac{5}{2}} \tau_i \left(\frac{T_i}{m_i}\right)^{1/2} \frac{1}{T_i} \nabla T_i(t-t'), \qquad (4.154)$$

for the ion heat flux, and

$$h_{rs}^{i(2)}(t) = \sum_{q=2,4} G^{(pq)}(t) h_r^{i(q)}(0) - \nu_i \int_0^t dt' \, G^{(22)}(t') \frac{\tau_i}{\sqrt{2}} W_{ij}(t-t'), \qquad (4.155)$$

for the ion stress tensor moments. Substituting these expressions (4.154-4.155) into the hydrodynamical equations using the definitions (4.133) and (4.134), will achieve closure as the heat flux and pressure tensor are functionals of the conserved moments, n_α $\vec{u}_\alpha$ and T_α. The present form of (4.154) and (4.155) would give rise to integro-differential equations for the evolution of the temperature and velocity, in which the rate of change of the temperature would depend on its previous history, as well as the initial conditions for the heat flux $\vec{q}_i(0)$ and stress $\hat{\sigma}_i(0)$. These are features not present in classical hydrodynamics. Based on the assumption that the hydrodynamic scales are much longer than the scale upon which collisions occur (4.12, 4.13) i.e. $\delta \ll 1$, one should look for the asymptotic solution to (4.154) and (4.155) for times t much longer than the ion collision time, τ_i. For $\delta \ll 1$ the propagator $G^{(pq)}(t)$ can be shown to decay exponentially within a few collision times $t \sim 7\tau_i$. The initial values for the heat flux and stress in (4.154) and (4.155) then disappear, the upper limits of the integrals in (4.154) and (4.155) can be replaced by infinity (adding essentially zero), and the sources $\nabla T_i(t-t')$, $W_{ij}(t-t')$ expanded as a Taylor series about time t. Ignoring all but the first term in the Taylor series $\nabla T_i(t-t') \sim \nabla T_i(t)$, $W_{ij}(t-t') \sim W_{ij}(t)$ gives familiar classical expressions that are local in both space and time. In the 21M approximation, these agree very well with those obtained by Braginskii. In fact the 21M method is equivalent to the Chapman-Enskog method with two Laguerre-Sonine polynomials [41]. The 13M approximation (which is obtained from the 21M

approximation by neglecting the moments $h_r^{i(5)}$ and $h_{rs}^{i(4)}$) is somewhat poorer, with deviations of up to two times from the Chapman-Enskog values, especially for the ion thermal conductivity and for large Z [41].

4.6.2 Nonlocal ion transport

It has been shown by Bychenkov *et al.* that retaining the temporal nonlocality inherent in (4.154, 4.155) allows for a valid description of ion transport in plasmas with large ZT_e/T_i [64]. This can be attributed to the fact that temporal nonlocality is more important than spatial nonlocality, as in ion acoustic waves, the phase velocity $c_s \sim \sqrt{ZT_e/T_i}$ exceeds the ion thermal velocity $v_{Ti} = \sqrt{T_i/m_i}$ by $\sqrt{ZT_e/T_i}$ and the frequency ω exceeds the ion-ion collision frequency ν_i, $\omega \gg \{kv_{Ti}, \nu_i, \nu_{ei}m_e/m_i\}$. In the nonlocal theory we assume that the transients arising from the initial preparation of the system have disappeared leaving the temporally nonlocal expressions for the heat flux

$$q_r^i(\vec{x},t) = -\frac{\sqrt{5\pi}}{2}\frac{n_i T_i \tau_i}{m_i}\left[\nu_i \int_0^t dt'\, G^{(33)}(t')\frac{\partial}{\partial x_r} T_i(t-t')\right], \tag{4.156}$$

and stress tensor

$$\pi_{rs}^i(\vec{x},t) = -n_i T_i \tau_i \left[\nu_i \int_0^t dt'\, G^{(22)}(t') W_{rs}(t-t')\right], \tag{4.157}$$

It is convenient to work in Fourier space, as the closure relations are then simple algebraic expressions rather than convolutions,

$$-i\omega\delta\vec{u}_i = -\frac{Ze}{m_i} i\vec{k}\delta\phi - \frac{i\vec{k}}{m_i n_i}(\delta n_i T_i + \delta T_i n_i) + \frac{1}{m_i n_i} i\vec{k}\cdot\delta\hat{\sigma}_i, \tag{4.158}$$

$$-i\omega\delta T_i = -\frac{2}{3} T_i\, i\vec{k}\cdot\delta\vec{u}_i - \frac{2}{3n_i}\frac{\partial}{\partial\vec{x}}\cdot\delta\vec{q}_i. \tag{4.159}$$

The friction terms in (4.158, 4.159) have been ignored, since for the low frequencies, the plasma motion is quasineutral $\delta n_e \sim \delta n_i$ and current free $\delta\vec{j} \sim 0$. Also, in the ion temperature equation (4.159) the heat generation term is also ignored as this is a very

slow process, negligible on the time scales of interest $\omega \gg \nu_{ei} m_e/m_i$. The frequency dependent closure for the ion stress tensor $\delta\sigma_i = (\vec{k} \cdot \delta\hat{\sigma}_i \cdot \vec{k})/k^2$ reads

$$\delta\sigma_i = \frac{4}{3} i\eta_i(\omega)\, \vec{k} \cdot \delta\vec{u}_i, \tag{4.160}$$

where the ion viscosity is given by the simple expression,

$$\eta_i = n_i T_i \tau_i\, \tilde{\eta}, \qquad \tilde{\eta}_i = \frac{i\nu_i(\omega + 1.46\, i\nu_i)}{(\omega + 1.20\, i\nu_i)(\omega + 1.46\, i\nu_i) + 0.23\, \nu_i^2}. \tag{4.161}$$

The numerical coefficients in the last expression (4.161) come from the from the matrix elements of the ion-ion collision operator (4.147). In the static limit $\omega \to 0$ $\tilde{\eta}_i = 0.96$, which agrees with Braginskii's result [1] for the ion viscosity. The closure for the ion heat flux may be written

$$\delta\vec{q}_i = -i\vec{k}\, \kappa_i(\omega) \delta T_i \tag{4.162}$$

in terms of the frequency dependent ion thermal conductivity

$$\kappa_i = \frac{5}{2} \frac{n_i T_i}{m_i} \tau_i\, \tilde{\kappa}_i, \qquad \tilde{\kappa}_i = \frac{i\nu_i(\omega + 1.29\, i\nu_i)}{(\omega + 0.80\, i\nu_i)(\omega + 1.29\, i\nu_i) + 0.21\, \nu_i^2}. \tag{4.163}$$

Again the numerical coefficients come from the matrix elements (4.146). The static limit for the thermal conductivity $\omega \to 0$, $\tilde{\kappa} = 1.56$ is also in agreement with Braginskii's result [1].

The set of equations (4.158, 4.159) together with the closure relations (4.160-4.163) constitute a valid description of ion dynamics where $ZT_e/T_i \gg 1$, in which case Landau damping is an unimportant mechanism [64].

Chapter 5

ION ACOUSTIC FLUCTUATIONS IN LASER PLASMAS

5.1 Introduction

The creation of hot dense plasmas with lasers is an essential feature of X-ray lasing schemes as well as inertial fusion experiments. In such plasmas Thomson scattering is useful for both characterization, which is necessary in order to calibrate and verify computer simulations, and also in the investigation of basic plasma physics. It has recently become possible for Thomson scattering to measure ion acoustic wave features such as damping and phase velocity in laser plasmas which allows ionization and temperature to be time resolved [35, 36]. Advances in the understanding of scattering instabilities have been made possible by Thomson scattering from enhanced levels of plasma fluctuations (cf. e.g. [29, 30]). Furthermore, Thomson scattering has been used as a tool for understanding basic plasma physics close to thermodynamic equilibrium. For example, both branches of the ion acoustic dispersion relation have been directly observed in a plasma with two ion species [31], and the ion plasma wave dispersion relation has been verified for the first time [32]. The utility of Thomson scattering, of which the above are examples, can be further enhanced when used in conjunction with better theoretical models.

The cross section for the Thomson scattering of laser light from plasmas is determined by $S(k,\omega)$, the Fourier transform of the electron density autocorrelation

function. This is well known in both the strongly collisional and collisionless limits, whilst the wide intermediate (weakly collisional) region of importance to laser plasmas has not yet been addressed. In Chapter 3 we have demonstrated that the two point correlation function of the phase space fluctuation $\langle \delta f^\alpha(\vec{x},\vec{v},t)\delta f^\beta(\vec{x}_0,\vec{v}_0,t_0)\rangle$ obeys a linearized version of the kinetic equation for the one particle distribution function $f^\alpha(\vec{x},\vec{v},t)$ in the $\vec{x}$, $\vec{v}$, t variables. This is a kinetic version of Onsager's "regression of fluctuations" [46] whereby fluctuations evolve from their initial values according to the equations of linearized hydrodynamics. Indeed it can be shown that this kinetic description reduces to Onsager's prescription in the hydrodynamic regime ($l_\alpha/L_H \ll 1, \nu_\alpha\tau_H \gg 1$) by a modification of the Chapman–Enskog method (cf. e.g. [18]). Here l_α and ν_α are the collisional mean free path and collision frequency of species α respectively, and L_H, τ_H are the length and times scales for the evolution of the fluctuating hydrodynamic variables. The derivation of Onsager's method from kinetic theory can be used to justify the validity of the method not only for thermodynamic equilibrium, but also for fluctuations about some nonequilibrium background state, that may for example support a heat flux. This will be made the subject of Chapter 6.

We will further extend the method's validity outside of the usual hydrodynamic regime by making use of hydrodynamic-like models that capture kinetic effects.

We will analyze in detail two cases of our general expression for the dynamic form factor, $S(k,\omega)$: the ion weakly collisional case where ion viscosity (modified by finite frequency) is important together with collisionless electron Landau damping, and the weakly collisional electron case in which the ions are collisional and the electron transport nonlocal. In the ion weakly collisional case we present an analytic expression for $S(k,\omega)$ that describes the effect ion-ion collisions have on the position and width of the ion acoustic peaks in the scattered spectrum. We discuss the relevence of these findings to some experiments reported in the literature [35, 36]. We

will also outline the range of parameters in which ion collisional effects are important and the usual collisionless theory of $S(k,\omega)$ is inadequate [21]. Our theory of $S(k,\omega)$ also predicts the correct line shape for plasmas with weakly collisional electrons that are commonly encountered in laser plasma interaction experiments. The height of the ion acoustic peaks is determined by the damping of ion acoustic waves. Since this damping depends on plasma transport properties, in particular electron thermal conductivity, we propose that the nonlocality of heat transport may be inferred from the scattered spectra. We assert that these descriptions are not only correct for hydrodynamic fluctuations, but also for fluctuations whose ratio of wavelength to mean free path is arbitrary. Comparison of our results with the standard collisionless cases will also be used in order to justify our method.

5.2 Theory of low frequency fluctuations

5.2.1 The closure problem

Laser plasmas are quite often non-isothermal as a result of inverse bremsstrahlung heating that preferentially heats the electrons, $T_e \gtrsim T_i$. The ionization can also be large especially for heavy elements such as gold, $Z \gg 1$. Therefore in many experiments there exists a separation in scale between the electron and ion collisionalities expressed by the relation $l_{ei} = (ZT_e/T_i)^2 l_i/\sqrt{2}$, where l_{ei} and l_i are the electron-ion and ion-ion collisional mean free paths, $l_{ei} = v_{Te}/\nu_{ei}$ and $l_i = v_{Ti}/\nu_i$. Here we have adopted the usual definition of collision frequencies,

$$\nu_{ei} = \frac{4\sqrt{2\pi}Ze^4 n_e \Lambda_e}{3\sqrt{m_e}T_e^{3/2}}, \quad \text{and} \quad \nu_i = \frac{4\sqrt{\pi}Z^4 e^4 n_i \Lambda_i}{3\sqrt{m_i}T_i^{3/2}}, \tag{5.1}$$

where Λ_α are the Coulomb logarithms. Considering an ion acoustic fluctuation in the plasma with a wave vector k, and the separation $l_{ei} \gg l_i$ we consider the following

possibilities:

$$kl_i,\ kl_{ei} \ll 1 \quad \text{strongly collisional case (Braginskii)} \tag{5.2}$$

$$kl_i \ll 1,\ kl_{ei} \sim 1 \quad \text{weakly collisional electrons} \tag{5.3}$$

$$kl_i \sim 1,\ kl_{ei} \gg 1 \quad \text{weakly collisional ions} \tag{5.4}$$

$$kl_i,\ kl_{ei} \gg 1 \quad \text{collisionless case (Vlasov)} \tag{5.5}$$

In the first case (5.2), the linearized fluid equations of Braginskii [1] correctly describe the evolution of the fluctuations and ion acoustic damping is determined in terms of the classical transport coefficients of thermal conduction and viscosity. In the last case (5.5), the collisionless, linearized Vlasov descriptions of fluctuations is appropriate (3.48) and damping is then due to wave-particle resonance (Landau damping) which depends on the form of the distribution function in velocity space at the phase velocity of the wave. These two cases are well known, but as yet the two intermediate cases are not, and have no self-consistent description. They are however very important because with typical k vectors and conditions in laser plasma experiments one invariably finds oneself in either of the two intermediate cases. See, for example the experiments of La Fontaine *et al.* [35, 36], and Tracy *et al.* [34].

In order to describe the electron weakly collisional regime $kl_{ei} \sim 1$ (5.3) we will make use of the nonlocal theory of electron transport that has been described in Section 4.4 of Chapter 4. This theory is based upon the solution to the linearized electron Fokker-Planck (Landau) equation by a Legendre polynomial expansion, $\delta f^e(k, \vec{v}, \omega) = \sum_l \delta f_l(v) P_l(\cos\theta)$. In this work Bychenkov *et al.* have been able to express the first Legendre coefficient δf_1 in terms of the hydrodynamic variables $\vec{E}^*$, the effective electric field, T_e, the electron temperature, and $\vec{u}_i$, the ion velocity in a way reminiscent of the Chapman–Enskog development, but without the restrictions

of strong collisions. Since the phase space fluctuation δf^a (3.8) obeys the same equations as the perturbation of the distribution function in the work of Bychenkov *et al.* [8] we may here interpret the δf^a to be the phase space fluctuation. We emphasize that this theory has a domain of validity beyond that of classical transport theory.

In describing the ion weakly collisional case (5.4) the usual classical transport for ions is not sufficient. To address this problem the analytic method of expansion of the ion kinetic equation in tensor Hermite polynomials described in Section 4.5 of Chapter 4 is used. This method has been shown in Ref. [64] to correctly describe ion acoustic wave properties in the limit $\omega \gg k v_{Ti}$. The damping of ion waves is in agreement with Braginskii in the collisional limit, $\omega \ll \nu_i$, and also agrees with Fokker-Planck solutions in the intermediate regime of collisionality, $\omega \gtrsim \nu_i$, for large ZT_e/T_i [64].

We will use the above closures together with the linearization of (3.58)-(3.60) in order to cover both cases (5.3) and (5.4) of weakly collisional plasmas that are often encountered experimentally [35, 36, 31, 32].

5.2.2 Nonlocal closure

We start by writing the system of linearized moment equations for the fluctuating hydrodynamical quantities δn_α, $\delta \vec{u}_\alpha$, δT_α, obtained from the kinetic equation for the phase space particle density fluctuation $\delta f^\alpha(\vec{x}, \vec{v}, t)$, $(\alpha = e, i)$ as prescribed by (3.58)-(3.60). Since ions are predominantly responsible for momentum transport, we write the ion momentum equation with the viscous term but neglect the ion thermal transport effect and the electron-ion energy exchange $\delta Q \to 0$ in (3.60) as these terms are small in comparison to momentum transport described by the viscosity tensor $\delta\hat{\sigma}^i$, particularly for plasmas with $ZT_e/T_i \gg 1$. Whilst it is the ions that carry the momentum, it is the electrons that are responsible for the heat transport.

We also make approximations pertinent to low frequency fluctuations. We assume the quasineutral limit, $\delta n_e \approx Z\delta n_i$, so that we restrict ourselves to long wavelength perturbations, $k\lambda_{De} \ll 1$ where λ_{De} is the electron Debye length.

$$\frac{\partial \delta n_\alpha}{\partial t} + n_\alpha \frac{\partial}{\partial \vec{x}} \cdot \delta \vec{u}_\alpha = 0, \tag{5.6}$$

$$\frac{\partial \delta \vec{u}_i}{\partial t} = -\frac{Ze}{m_i} i\vec{k}\delta\phi - \frac{i\vec{k}}{m_i n_i}(\delta n_i T_i + \delta T_i n_i) + \frac{1}{m_i n_i} i\vec{k} \cdot \delta\hat{\sigma}_i + \frac{1}{m_i n_i}\delta R_{ie}, \tag{5.7}$$

$$\frac{\partial \delta T_i}{\partial t} + \frac{2}{3} T_i i\vec{k} \cdot \delta \vec{u}_i = 0, \tag{5.8}$$

$$\frac{\partial \delta T_e}{\partial t} + \frac{2}{3n_e}\frac{\partial}{\partial \vec{x}} \cdot \delta \vec{q}_e + \frac{2}{3} T_e \frac{\partial}{\partial \vec{x}} \cdot \delta \vec{u}_e = 0, \tag{5.9}$$

where $\delta \vec{u}_e = \delta \vec{u}_i - \delta \vec{j}/en_e$. The phase space fluctuation δf^e is solved for in terms of the hydrodynamic moments $\delta \vec{u}_i$, δn_e, δT_e and the potential $\delta\phi$ as described in 4, where we found the closure relations,

$$\delta \vec{j} = \sigma \delta \vec{E}^* + \alpha i\vec{k}\delta T_e + \beta_j e n_e \delta \vec{u}_i, \tag{5.10}$$

$$\delta \vec{q}_e = -\alpha T_e \delta \vec{E}^* - \chi i\vec{k}\delta T_e - \beta_q n_e T_e \delta \vec{u}_i, \tag{5.11}$$

$$\delta \vec{R}_{ie} = -(1-\beta_j) n_e e \delta \vec{E}^* + \beta_q n_e i\vec{k}\delta T_e - \beta_r m_e n_e \nu_{ei} \delta \vec{u}_i, \tag{5.12}$$

where $\delta \vec{E}^* = -i\vec{k}\delta\phi + i\vec{k}/en_e(\delta n_e T_e + n_e \delta T_e)$ is the effective electric field usually introduced in classical transport theory [41]. These closure relations are written in Fourier space as the transport coefficients are all k and ω dependent. In real space the closure relations will become convolution operators. Since we are concerned with quasineutral fluctuations the relation $\delta \vec{j} = 0$ (5.10) gives the following expression for the heat flux (5.11) on eliminating the electric field $\vec{E}^*$,

$$\delta \vec{q}_e = -\kappa i\vec{k}\delta T_e - \beta n_e T_e \delta \vec{u}_i, \tag{5.13}$$

where $\kappa = \chi - \alpha^2 T_e/\sigma$, and $\beta = \beta_q - e\alpha\beta_j/\sigma$. The transport coefficients in this theory are α, the thermocurrent coefficient, χ, the thermal conductivity, σ, the electrical conductivity and the new transport coefficients β_q, β_j and β_r that are related to the ion flow. All the coefficients are dependent on the ionization, Z, k, and ω. Rather than

tabulate numerical values for the coefficients, JM has made available upon request a Fortran code that calculates all the necessary transport coefficients and is detailed in Appendix C.1.

In order to close the set (5.6)-(5.9) all that remains is the closure for the ion stress tensor, $\delta\sigma_i$, that is valid for case (5.4). From Section 4.5 the Grad 21M closure for the longitudinal part of the viscosity tensor,

$$\delta\sigma_i = \frac{\vec{k}\cdot\delta\hat{\sigma}_i\cdot\vec{k}}{k^2} = \frac{4}{3}\frac{n_iT_i}{\nu_i}\tilde{\eta}_i(\omega)\, i\vec{k}\cdot\delta\vec{u}_i, \tag{5.14}$$

results in a frequency dependent ion viscosity that has both a real and imaginary part,

$$\tilde{\eta} = \frac{i\nu_i(\omega + 1.46i\nu_i)}{(\omega + 1.20i\nu_i)(\omega + 1.46i\nu_i) + 0.23\nu_i^2}. \tag{5.15}$$

In previous work [64] that was concerned with ion acoustic damping it has been demonstrated that the real part of (5.15) produces the correct damping of ion acoustic waves with a smooth transition from the strongly collisional Braginskii limit $\gamma_i \approx 0.64k^2v_{Ti}^2/\nu_i$ to the saturated Rukhadze limit [70] where $\gamma_i \approx 0.8\nu_iT_i/ZT_e$. It also compares well to the Fokker–Planck simulations of [71, 73] in the intermediate region of collisionality. The imaginary part affects the transition from the adiabatic to the isothermal phase speed as ω exceeds the ion-ion collision frequency ν_i. We now set out our generalized version of Onsager's "regression of fluctuations" that was outlined in Chapter 4 using the closures $\delta\vec{j} = 0$, (5.10), (5.12) and (5.13) to the linear hydrodynamic moment equations (5.6)-(5.9).

5.2.3 Correlations of the fluctuating hydrodynamic variables

In order to be able to calculate hydrodynamic correlations we take the Laplace transform in time of the set (5.6)-(5.9) and the Fourier transform in space,

$$-i\omega\delta n_e \;+\; n_e i\vec{k}\cdot\delta\vec{u} = \delta n_e(0), \tag{5.16}$$

$$
\begin{aligned}
-i\omega\delta\vec{u} &= -\frac{Ze}{m_i}i\vec{k}\delta\phi - \frac{i\vec{k}}{m_i n_i}(n_i\delta T_i + T_i\delta n_i) \\
&\quad + \frac{1}{m_i n_i}i\vec{k}\cdot\delta\hat{\sigma}_i + \frac{1}{m_i n_i}\delta\vec{R}_{ie} + \delta\vec{u}(0), \qquad (5.17)\\
-i\omega\delta T_e &+ \frac{2}{3n_e}i\vec{k}\cdot\delta\vec{q}_e + \frac{2}{3}T_e i\vec{k}\cdot\delta\vec{u} = \delta T_e(0), \qquad (5.18)\\
-i\omega\delta T_i &+ \frac{2}{3}T_i i\vec{k}\cdot\delta\vec{u} = \delta T_i(0). \qquad (5.19)
\end{aligned}
$$

Here δu is the hydrodynamic velocity perturbation which is the same for ions and electrons since we consider quasineutral perturbations, $\delta j = 0$. Equations (5.16)-(5.19) describe the evolution of the fluctuating hydrodynamic variables from their initial values at time $t = 0$. This is sufficient for the calculation of the correlations of any of the hydrodynamic variables by following the prescription outlined in Chapter 4. For example, $\langle \delta T_e \delta T_e^*(0)\rangle$ may be formed by solving the set (5.16)-(5.19) (with the appropriate closure) for the transformed δT_e in terms of the initial fluctuations, multiplying by $\delta T_e^*(0)$ and then ensemble averaging. The solution is then given in terms of the initial correlations which are known (3.61). The initial correlations are simplified as the different hydrodynamic variables are independent of each other by virtue of the initial condition (3.61). The Fourier transform of the correlation function, $\langle \delta T_e \delta T_e^*\rangle$ is then related to the Laplace transform by $\langle \delta T_e \delta T_e^*\rangle = 2\mathrm{Re}\langle \delta T_e \delta T_e^*(0)\rangle$, as explained in Chapter 4 (3.56). We now specialize this to the calculation of $S(k,\omega) = \langle \delta n_e \delta n_e^*\rangle / n_e$ because of its usefulness in determining the cross section for Thomson scattering.

5.2.4 Calculation of the dynamic form factor

In solving (5.16)-(5.19), we will ignore the time derivative in the electron heat equation (5.18) as it is consistent with our desire to describe isothermal ion acoustic fluctuations ($\omega \sim kc_s$ and $kl_{ei} \gg c_s/v_{Te}$), where c_s is the cold ion sound speed $c_s = \sqrt{ZT_e/m_i}$. Also, in calculating $S(k,\omega)$ we can neglect all initial conditions except $\delta n_e(0)$, since all others are uncorrelated with the choice of initial conditions (3.61). The condition

of zero current, $\delta\vec{j} = 0$ (5.10) gives an expression for the fluctuating potential

$$i\vec{k}\delta\phi = \frac{i\vec{k}}{en_e}(\delta n_e T_e + \delta T_e n_e) + \frac{\alpha}{\sigma} i\vec{k}\delta T_e + \frac{\beta_j}{\sigma} en_e\delta\vec{u}. \tag{5.20}$$

This can be used to eliminate the potential term in the ion momentum equation (5.17) and also in the expression for the friction $\delta\vec{R}_{ie}$ (5.12),

$$\delta\vec{R}_{ie} = n_e\left(\beta + \frac{e\alpha}{\sigma}\right) i\vec{k}\delta T_e + (1-\beta_j)\beta_j\frac{e^2n_e^2}{\sigma}\delta\vec{u} - \beta_r m_e n_e \nu_{ei}\delta\vec{u}. \tag{5.21}$$

With the closure (5.13) for the electron heat flux, the electron temperature equation (5.18) can be solved for δT_e,

$$\delta T_e = -\frac{n_e T_e}{k^2\kappa}(1-\beta)i\vec{k}\cdot\delta\vec{u}. \tag{5.22}$$

On substituting (5.20), (5.21) and (5.22) together with the expression (5.15) for the ion viscosity and ion temperature into the ion momentum equation (5.17) and after using the continuity equation (5.17) in order to express the velocity in terms of density, the density perturbation is expressed in terms of the initial perturbation,

$$D(k,\omega)\delta n_e(k,\omega) = \delta n_e(0), \quad \text{where} \quad D = -i\omega + i\frac{k^2(c_s^2 + v_{Ti}^2)}{\omega - \Delta + 2i\gamma_a}. \tag{5.23}$$

Here D is the dispersion equation for ion acoustic waves,

$$\gamma_a = \frac{n_e c_s^2(1-\beta)^2}{2\kappa} + \frac{n_e e^2 c_s^2}{2\sigma T_e}\beta_j^2 + \beta_r\nu_{ei}\frac{c_s^2}{2v_{Te}^2} + \gamma_i, \tag{5.24}$$

is the damping rate with the ion viscous contribution,

$$\gamma_i = \frac{2}{3}\frac{k^2v_{Ti}^2}{\nu_i}\,\mathrm{Re}\,\tilde{\eta} = k^2v_{Ti}^2\frac{\nu_i(1.49\,\nu_i^2 + 0.80\,\omega^2)}{\omega^4 + 4.05\,\nu_i^2\omega^2 + 2.33\,\nu_i^4}, \tag{5.25}$$

and

$$\Delta = \frac{2}{3}\frac{k^2v_{Ti}^2}{\omega} + \frac{4}{3}\frac{k^2v_{Ti}^2}{\nu_i}\,\mathrm{Im}\,\tilde{\eta} \tag{5.26}$$

accounts for ion contribution to the ion acoustic wave dispersion due to ion viscosity and heating. Using (5.23) we can express $\langle\delta n_e\delta n_e^*(0)\rangle/n_e$ in terms of the initial

correlations as given by (3.61) $\langle \delta n_e(0) \delta n_e^*(0) \rangle / n_e = 1$. From the relation $S(k,\omega) = 2\mathrm{Re}\langle \delta n_e \delta n_e^*(0) \rangle / n_e$ the dynamic form factor is determined:

$$S(k,\omega) = \frac{4\,k^2(c_s^2 + v_{Ti}^2)\gamma_a}{(\omega^2 - k^2 v_s^2)^2 + 4\,\omega^2 \gamma_a^2}, \tag{5.27}$$

where we have introduced the definitions,

$$v_s = \sqrt{c_s^2 + \Gamma_i v_{Ti}^2}, \quad \text{and} \quad \Gamma_i = \frac{5}{3} + \frac{4}{3}\frac{\omega}{\nu_i}\,\mathrm{Im}\,\tilde{\eta} = \frac{9\,\omega^4 + 29.7\,\omega^2\nu_i^2 + 11.7\,\nu_i^4}{3\,(\omega^4 + 4.05\,\omega^2\nu_i^2 + 2.33\,\nu_i^4)}, \tag{5.28}$$

for the ion acoustic group velocity and ion specific heat ratio.

5.3 Applications

5.3.1 Application of the nonlocal theory in the limit of collisional ions

There are two main issues that can be addressed concerning the application of our theory for the ion acoustic feature in the Thomson scattered spectrum in this regime (5.4). The first is ion acoustic damping, which determines the height of the ion acoustic peaks. In the intermediate regime of collisionality $kl_{ei} \sim 1$ the electron contribution to ion acoustic damping has been investigated both theoretically [74] and numerically [75], as it is important for stimulated scattering processes. The damping may be calculated from the theory comprising (5.6)-(5.8) and (5.10)-(5.13) for the wavelengths $kl_{ei} \gg c_s/v_{Te}$, $kl_i \ll 1$,

$$\gamma_a = \frac{n_e c_s^2}{2}\left[\frac{(1-\beta)^2}{\kappa} + \frac{e^2\beta_j^2}{T_e\sigma} + \frac{\beta_r}{n_e v_{Te} l_{ei}}\right] + 0.64\frac{k^2 v_{Ti}^2}{\nu_i}, \tag{5.29}$$

and this compares well with the numerical solution to the Fokker-Planck kinetic equation [75] and the analytic theory [74]. It has the proper hydrodynamic form in the long wavelength limit $kl_{ei} \ll 1$ and takes the form of collisionless electron Landau

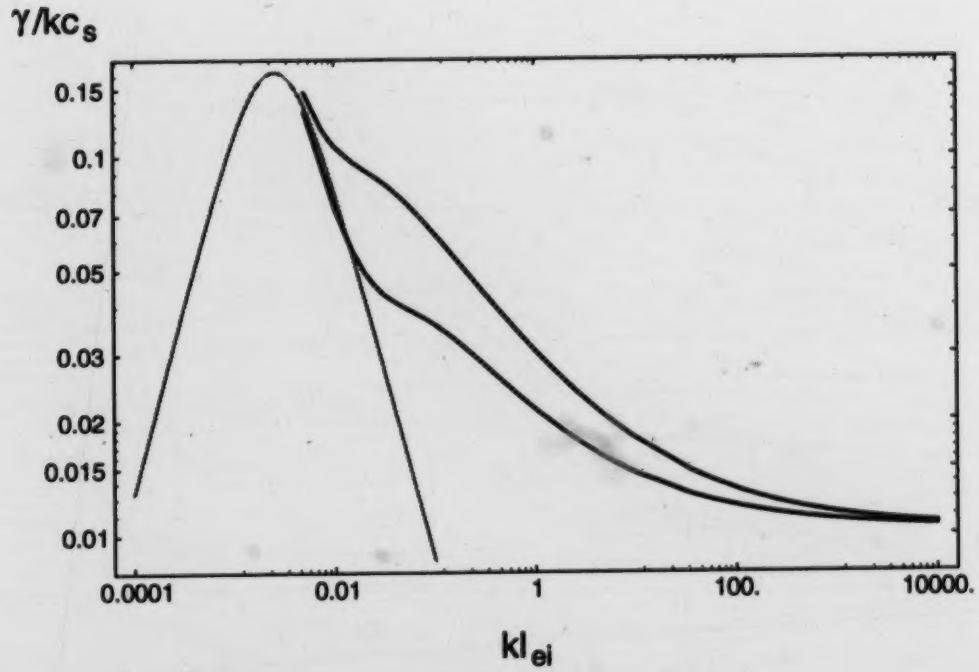

Figure 5.3.1: Electron part of ion acoustic damping, γ/kc_s, as a function of electron collisionality kl_{ei}. The grey curve shows the prediction of fluid theory with classical thermal conductivity. The solid lines show the damping from the analytic theory of [12] for $Z = 64$ (top) and $Z = 8$ (bottom).

damping in the short wavelength region $kl_{ei} \gg 1$. Figure (5.3.1) shows the damping as a function of electron collisionality kl_{ei} as predicted by (5.29). It is interesting to note that the deviation from classical Braginskii theory occurs early, whilst the wavelength is still hundreds of times larger than a mean free path. This will be reflected in $S(\vec{k},\omega)$ (5.27), whose form may be interpreted with the aid of Fig. (5.3.1).

The other issue is concerned with transport. The parameters of many laser plasma experiments fall in the regime of nonlocal transport as is demonstrated in Fig. (5.3.2) for the case of a high Z plasma. Since the line shape, or height of the ion acoustic peaks described by (5.27) is expressed in terms of transport coefficients, Thomson scattering may be used as a probe for this nonlocality. The probed k vector in the plasma is determined by $k = 2k_0 \sin(\theta/2)$ where k_0 is the wave vector of the incident probe beam and θ is the scattering angle chosen by the experimentalist. We propose a comparison between the spectrum for two (or more) different scattering angles. In this way the k dependence of the transport coefficients may be inferred. In choosing experimental parameters, Z should be sizeable for the validity of the nonlocal transport theory [8]. In Fig. (5.3.2) there are three lines that identify $\alpha = 1/k\lambda_{De}$ and the contours show electron-ion collisionality. Figures (5.3.3) and (5.3.4) show a comparison between the spectrum predicted by equation (5.27) and collisionless theory (2.18) for different scattering angles. In particular Fig. (5.3.4) shows how the effect of collisions alters the k dependence of the peak height from that expected from collisionless theory where fluctuations are only Landau damped. These parameters have been chosen to be close to those encountered experimentally, for example a gold plasma with the conditions $n_e = 0.5 \times 10^{21}$ cm^{-3}, $T_e = 1$ keV, $T_i = 500$ eV and $Z = 50$ and a 0.35 μm probe. Figure (5.3.5) shows a more collisional regime due to the use of a longer wavelength probe, which is compared to Braginskii theory. In this case the effect of changing the angle from 10° to 180° changes the collisionality of the probed ion acoustic fluctuation from $kl_{ei} \sim 0.01$ (where classical transport just starts

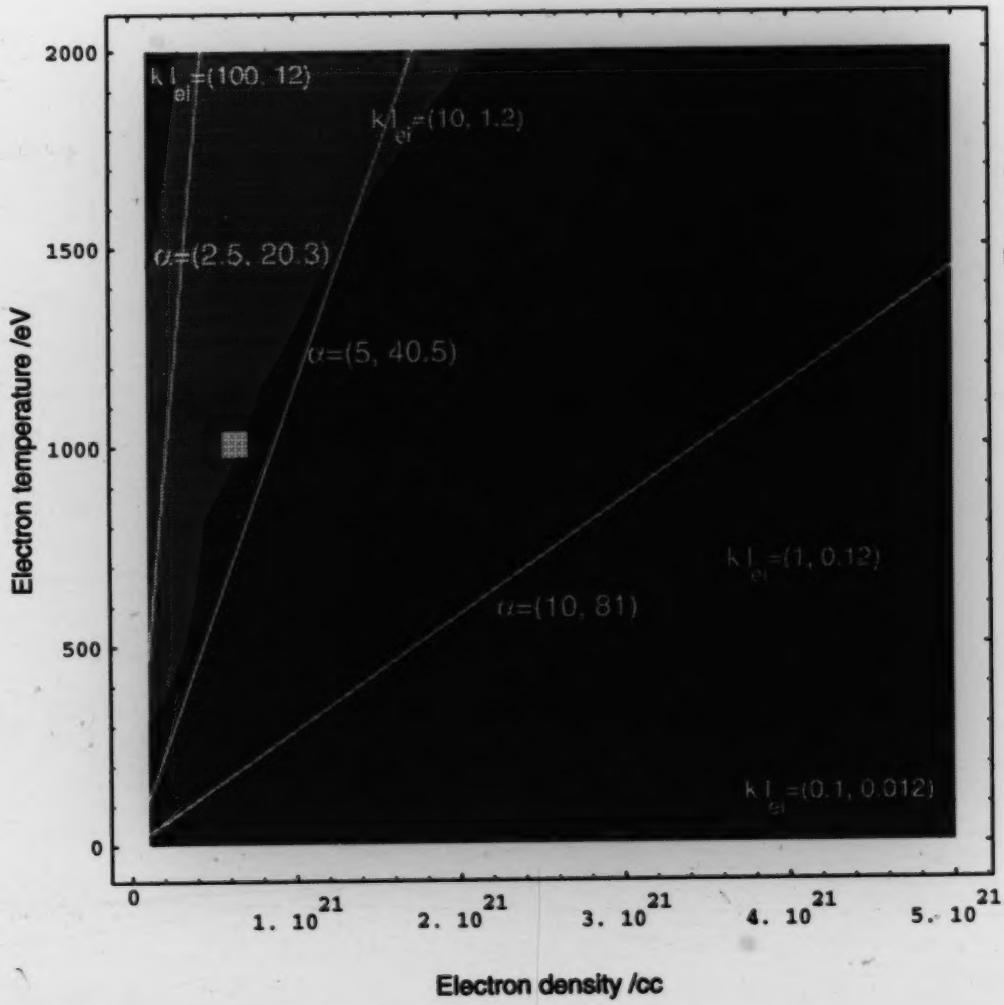

Figure 5.3.2: Parameter regime for gold plasma. The contour plot shows electron-ion collisionality kl_{ei} for scattering angles of 90° and 10°. The first number in the parentheses corresponds to 90° and the second to 10° for a 0.35 μm probe beam. Also shown is $\alpha = 1/k\lambda_{De}$ again for 90° and 10° scattering. The box shows the plasma parameters of Figs. 5.3.3 and 5.3.4.

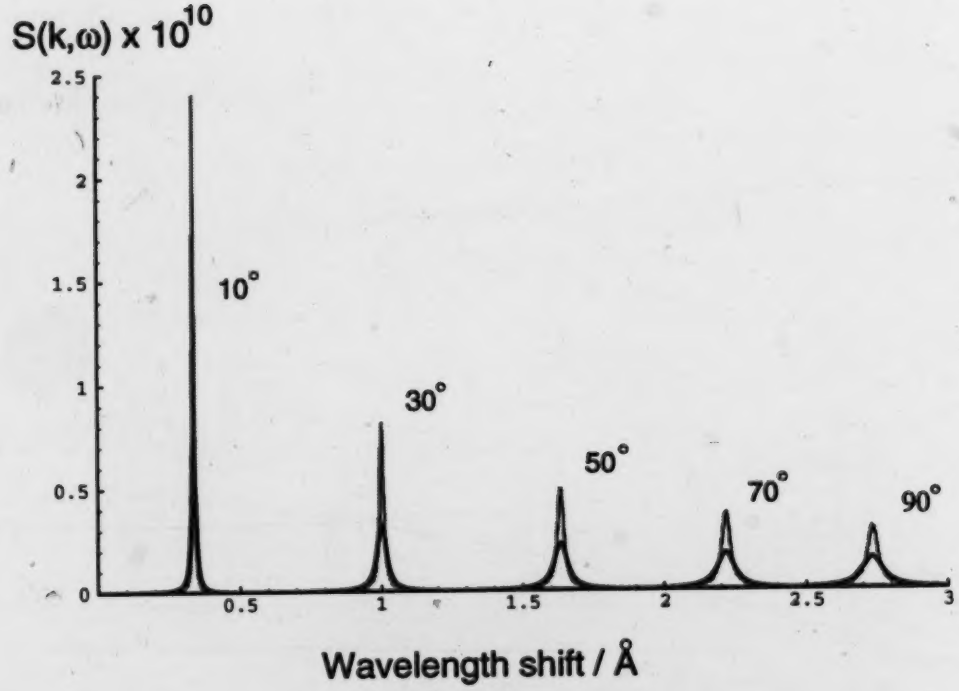

Figure 5.3.3: The dynamic form factor $S(k,\omega)$ for a weakly collisional gold plasma, $n_e = 0.5 \times 10^{21}$ cm^{-3}, $T_e = 1$ keV, $T_i = 0.5$ keV and $Z = 55$ for a 0.35 μm probe and different scattering angles. Grey is collisionless theory, solid is nonlocal theory.

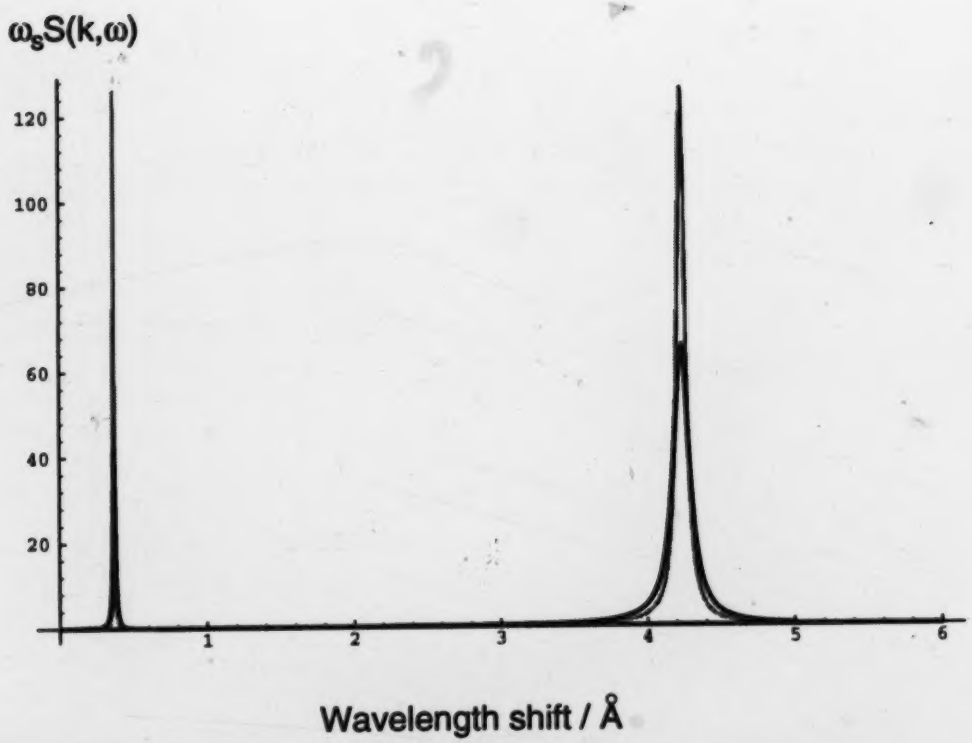

Figure 5.3.4: The dynamic factor $\omega_s S(k,\omega)$ normalized by the ion acoustic frequency ω_s for a gold plasma, $n_e = 0.5 \times 10^{21}$ cm^{-3}, $T_e = 1$ keV, $T_i = 0.5$ keV and $Z = 55$ for a 0.35 μm probe. This figure illustrates the difference between the Vlasov theory and the nonlocal theory for the scattering angles of 10° and 180°. Grey is collisionless theory, solid is nonlocal theory.

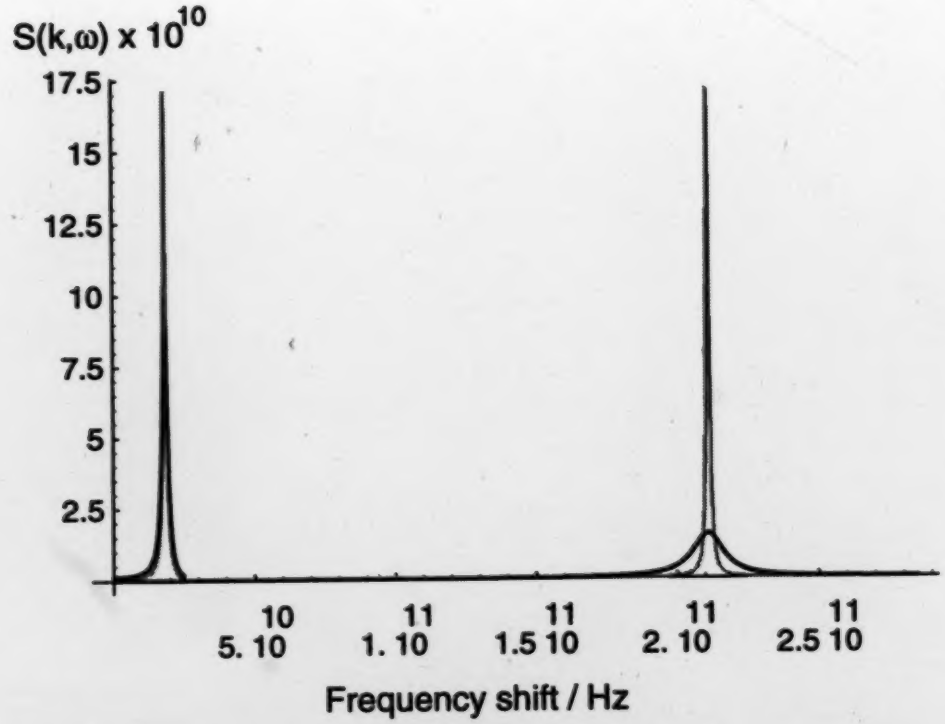

Figure 5.3.5: The dynamic form factor $S(k,\omega)$ for a more collisional gold plasma, $n_e = 2 \times 10^{21}$ cm^{-3}, $T_e = 1$ keV, $T_i = 0.5$ keV and $Z = 55$ for a 10.6 μm probe. The figure shows $S(k,\omega)$ at the angles of 10° (left) and 180° (right) and demonstrates the departure from classical hydrodynamics. Grey is Braginskii fluid equations with classical heat conductivity, solid is nonlocal theory.

to break down) to $kl_{ei} \sim 0.1$ (classical transport inadequate). This is an interesting regime as the main contribution in (5.27) to the scattering then comes from κ, the electron thermal conductivity. Investigation of the spectra in this regime could be used to test models of nonlocal thermal conductivity.

5.3.2 Application of the theory in the limit of collisionless electrons

In this regime of collisionless electrons $kl_{ei} \gg 1$, and semi-collisional ions, $kl_i \sim 1$, the damping γ_a (5.24) takes the form

$$\gamma_a = \sqrt{\frac{\pi}{8}} \frac{c_s}{v_{Te}} k v_s + \frac{2}{3} \frac{k^2 v_{Ti}^2}{\nu_i} \mathrm{Re}\, \tilde{\eta}, \tag{5.30}$$

which will be appropriate for discussing the experiments [35, 36]. The fluctuation spectrum (5.27) does not account for the entropy mode since we have neglected the ion thermal conductivity ($\omega \gg k v_{Ti}$). To assess this formula (5.27) we will compare the predictions to those of the collisionless theory for plasma parameters similar to those of the experiment due to La Fontaine *et al.* [36]. We define the range of plasma parameters for which ion-ion collisions can be important in determining the fluctuation spectra. Figure (5.3.6) shows the ion damping of ion acoustic waves as a function of ion-ion collisionality from Ref. [64]. Note that the effect of ion Landau damping, which is missing in (5.25) and (5.30), has been added phenomenologically in Fig. (5.3.6) according to [64]. For plasmas with $ZT_e/T_i > 40$ we have the situation where although the ion damping differs from the collisionless limit, the ion contribution is much less than that due to the electrons (electron Landau damping). We therefore identify the interesting range of parameters to be given by $8 \lesssim ZT_e/T_i \lesssim 40$. As an example, for $ZT_e/T_i = 16$ the ion damping is a few times smaller than the electron contribution in the collisionless limit, but with the addition of ion-ion collisions it becomes (for $kl_i \sim 0.2$) a few times larger than the electron (Landau damping) contribution, see Fig. (5.3.6). Ion acoustic waves will be more strongly damped in this regime than the collisionless theory would predict. This range of parameters has relevance to several recent experiments [35, 36, 34, 32].

In the experiments of La Fontaine *et al.* [35, 36], a difficulty is expressed in fitting the width of the observed spectra to the collisionless theory (2.18) (c.f. also

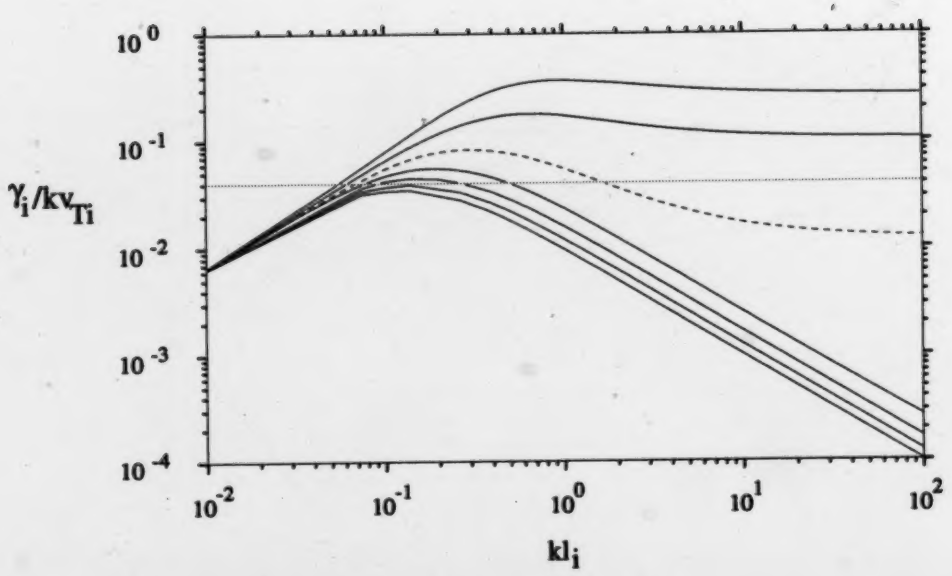

Figure 5.3.6: The dependence of γ_i/kv_{Ti}, the normalized ion part of ion acoustic damping, on ion collisionality kl_i. This damping includes an ion Landau damping contribution in addition to collisions. There are six curves plotted for the temperature ratios ZT_e/T_i of 4, 8, 16, 48, 64 and 80 (top to bottom). The dashed curve shows the electron Landau damping contribution for the case $ZT_e/T_i = 16$ and shows how the relative importance of the ions depends strongly on the ion collisionality.

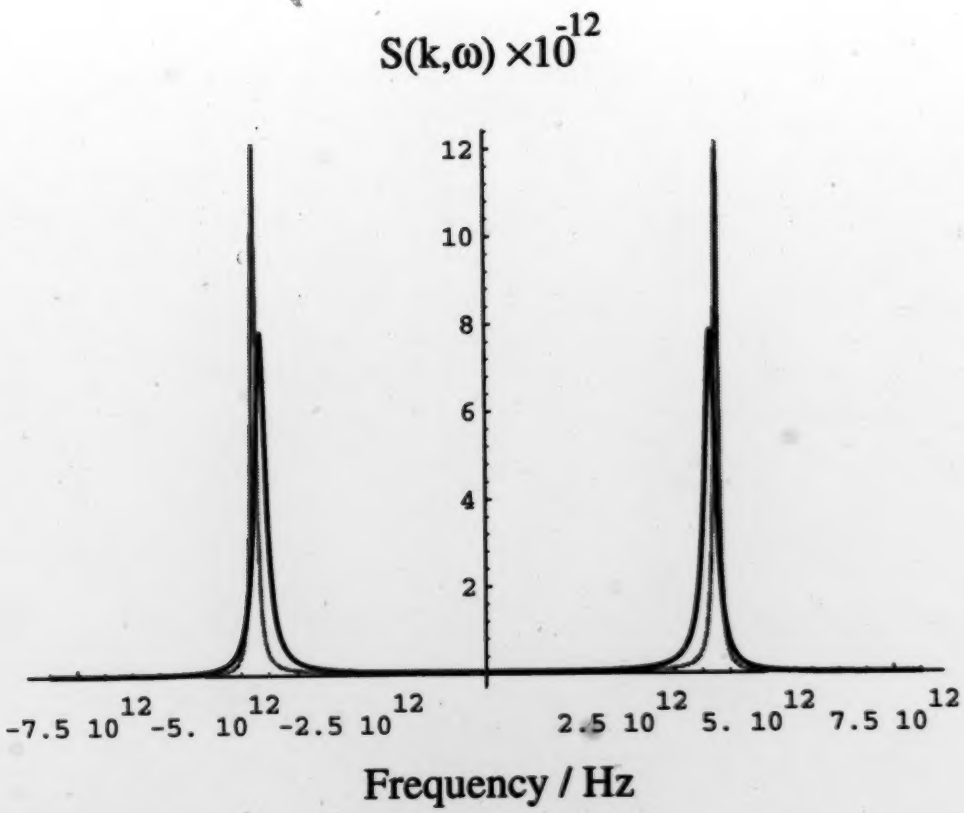

Figure 5.3.7: The ion acoustic peaks as predicted from Eq. (5.27) (solid lines) and from collisionless theory (light lines) for a carbon plasma, $ZT_e/T_i = 12$.

[37]). They note that this is possibly due to the effects of ion-ion collisions, and point out the need for further investigation. We address this situation for the plasma conditions of their experiment, see Table 5.3.1. Two cases considered are for carbon plasmas, in the first ZT_e/T_i is ~ 12 and in the latter ~ 8.6. The authors obtain T_i from the width of the peaks, as in the collisionless limit this is due to ion Landau damping. However, in this experiment the ions are not collisionless, $kl_i \sim 1$ and our ion acoustic peaks are twice as broad for the same T_i. A comparison of our spectra and the collisionless spectra appears in Figs. (5.3.7) and (5.3.8). The authors correctly point out that ion collisions can broaden the ion acoustic peaks. In addition,

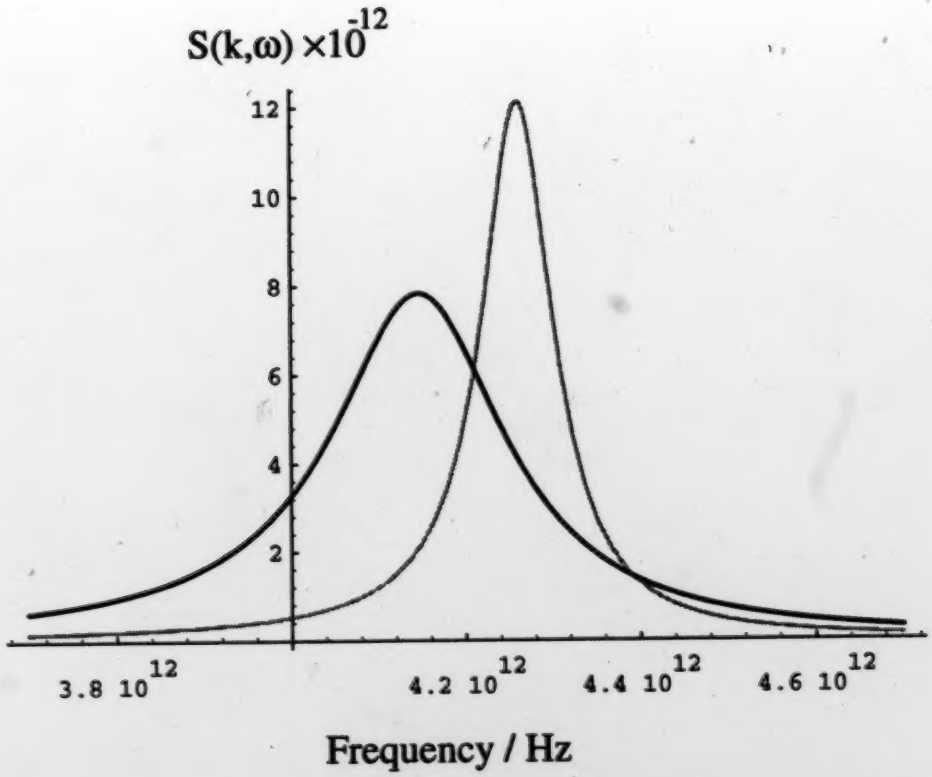

Figure 5.3.8: A close up of the ion acoustic peaks for the parameters of Fig. (5.3.7). The light line corresponds to the prediction of collisionless theory and solid line Eq. (5.27).

however, ion collisions modify the specific heat ratio and alter the phase speed of the ion acoustic mode. The phase speed, $v_s \approx c_s\sqrt{1+3T_i/ZT_e}$ in the collisionless limit, where the coefficient 3 corresponds to the isothermal specific heat ratio for ions. The effect of collisions is to reduce this coefficient towards 5/3 [73]. This effect is not large (a few percent) but it adds more error to the inferred electron temperatures (cf. Fig. (5.3.8)).

Ions	μ	Z	T_e/eV	T_i/eV	ZT_e/T_i	kl_i	kl_{ei}	γ_i/kv_{Ti}	γ_e/kv_{Ti}
Carbon	12	6	400	200	12	0.22	22	0.1	0.04
Carbon	12	6	500	350	8.6	0.65	34	0.18	0.03
Germanium	73	24	1300	500	62	0.02	49	0.02	0.07
Tantalum	181	45	1100	500	99	0.002	19	$\sim 10^{-3}$	0.07

Table 5.3.1: Parameters deduced from the experiments of [36]. Interferometry estimates n_e to be $\sim 5 \times 10^{20}\text{cm}^{-1}$. The scattering angle $\theta = 90°$ and the laser wavelength, $\lambda = 355\,\text{nm}$. The probed k vector in the plasma is $k \approx 0.25 \times 10^6\,\text{cm}^{-1}$ and $k/k_D \sim 0.2$. The ion temperatures for germanium and tantalum are not given in [36] and have been assigned by us.

Chapter 6

PLASMA WITH BACKGROUND GRADIENTS

6.1 Motivation

Experimental Thomson scattering spectra often show a peak height asymmetry in the ion feature corresponding to ion acoustic waves propagating in opposite directions [33, 34, 35, 36]. One possible source of this asymmetry in a relative drift between the electron and ion components [38]. Another, and the one considered here is the presence of a heat flux. A current free plasma carrying a heat flux will have a skewed "return current" distribution function in velocity space [74]. In the collisionless regime the sound wave propagating in the direction of the temperature gradient (and opposite to the direction of the heat flux) will have a reduced Landau damping rate, and can even become unstable for large enough heat fluxes $q \sim n_e T_e c_s$. The instability is well known in the collisionless regime [77], but recently it has been identified in the weakly collisional regime $kl_{ei} \sim 1$ also, where it has been attributed to the density dependence of the heat flux [64].

Here we present a generalization of the theory of the dynamic form factor from Chapter 5 to include plasma states that carry a background heat flux. It is considered worthwhile to obtain a theory describing such background states, due to the possibility of observing the enhanced fluctuations near the onset of the instability by a Thomson scattering measurement, which are often performed in the weakly collisional regime $kl_{ei} \sim 1$. Furthermore, it is anticipated that the difference in fluctuation spectra as predicted by the nonlocal model and collisionless Vlasov theory (2.18) will be

observable.

The calculation also demonstrates how the nonlocal formulation of electron transport described in Chapter 4 can be modified to include the effects of different background states and/or sources. For example, the effect of inverse bremsstrahlung and the ponderomotive force has been discussed in Ref. [80]. This goes at least some way towards resolving the problem dependence issue raised by Epperlein [52] and discussed in Section 4.3.

6.2 Background state of the plasma

We consider as a reference state a plasma with a temperature gradient supported by some unspecified external source which is stationary, and without hydrodynamical motion $\vec{u}_\alpha = 0$, $(\alpha = e, i)$. The spatial scale of the temperature inhomogeneity along the x-axis, $L_T = 1/(d\ln T_e/dx)$, is assumed to be sufficiently large $L_T > 100\, l_{ei}$ (4.64) so that the classical collisional transport theory can be applied to describe this reference state. The ions will be treated as a cold fluid in the background state, although the ion contribution to the dynamical form factor $S(\vec{k},\omega)$ will be accounted for in the Grad 21-moment approximation as previously derived in Section 4.5.

The electron distribution function $f_e(\vec{v},x,t)$ must satisfy the kinetic equation

$$\frac{\partial f_e}{\partial t} + v_x\frac{\partial f_e}{\partial x} + \frac{e}{m_e}\frac{\partial \phi}{\partial x}\frac{\partial f_e}{\partial v_x} = C_{ei}[f_e] + C_{ee}[f_e, f_e], \tag{6.1}$$

where $-e$ and m_e are the electron charge and mass and $\phi(x,t)$ is the electric potential. The electron-ion collision term

$$C_{ei}[f_e(\vec{v})] = \frac{1}{2}\nu_{ei}(v)\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f_e}{\partial \mu} \tag{6.2}$$

is written neglecting electron-ion energy exchange, where $\mu = v_x/v$ is the cosine of the angle between the electron velocity and x-axis, $\nu_{ei}(v) = 4\pi Z n_0 e^4 \Lambda_e/m_e^2 v^3$ is the

velocity dependent collision rate, and Λ_e is the Coulomb logarithm. The ion density n_i and velocity $\vec{u}_i$ are described by the Euler equations,

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \qquad \frac{\partial u_i}{\partial t} = -\frac{Ze}{m_i}\frac{\partial \phi}{\partial x} + \frac{1}{n_i m_i} R_{ie}, \tag{6.3}$$

where the direction of the gradient is in the x-direction and the friction force $R_{ie} = m_e \int d\vec{v}\; v_x \nu_{ei}(v) f_e(\vec{v})$. We are interested in temperature inhomogeneity scales much larger than a Debye length $L_T \gg \lambda_{De}$, hence we can approximate Poisson's equation by the quasineutrality condition $n_e \approx Z n_i$.

The electron distribution function is close to a local Maxwellian in the ion reference system,

$$F_0(x,v) = \frac{n_e}{(2\pi)^{3/2} v_{Te}^3} \exp\left(\frac{-v^2}{2v_{Te}^2}\right), \tag{6.4}$$

with a spatially inhomogeneous density $n_e(x)$ and temperature $T_e(x) = m_e v_{Te}^2$. The distribution function will consist of a Maxwellian part F_0 plus a small anisotropic correction F_∇ which is proportional to l_{ei}/L_T [49]

$$f_e(\vec{v}) = F_0 + \mu F_\nabla. \tag{6.5}$$

Substituting (6.5) into (6.1) gives an expression for the correction

$$F_\nabla(v) = \frac{v}{\nu_{ei}(v)}\left(\frac{e}{T_e}\frac{d\phi}{dx} - \frac{d}{dx}\ln F_0\right) F_0, \tag{6.6}$$

which comes from the electron-ion collision term. The contribution to the isotropic part of distribution function from the electron-electron collision term can be neglected as it is second order in the expansion parameter l_{ei}/L_T. The ion Euler equations (6.3) and condition of zero current $j = \int dv\; v^3 F_\nabla = 0$ provides the following relations,

$$n_e(x) T_e(x) = \text{const}, \qquad e\frac{d\phi}{dx} = \frac{3}{2}\frac{dT_e}{dx}. \tag{6.7}$$

The first of them shows the state is characterized by constant pressure and the second that the electric field $-\partial\phi/\partial x$ is balanced by the friction force coming from the

electron temperature gradient (4.59). Equations (6.6, 6.7) give an expression for the anisotropic part of the electron distribution function

$$F_\nabla(v) = \frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{l_{ei}}{L_T}\frac{v^4}{v_{Te}^4}\left(4 - \frac{v^2}{2v_{Te}^2}\right)F_0, \tag{6.8}$$

that is a solution to the electron kinetic equation (6.1) and the ion Euler equations (6.3). Figure 6.2.1 shows the perturbation (6.8) and the distribution function f_e (6.5)

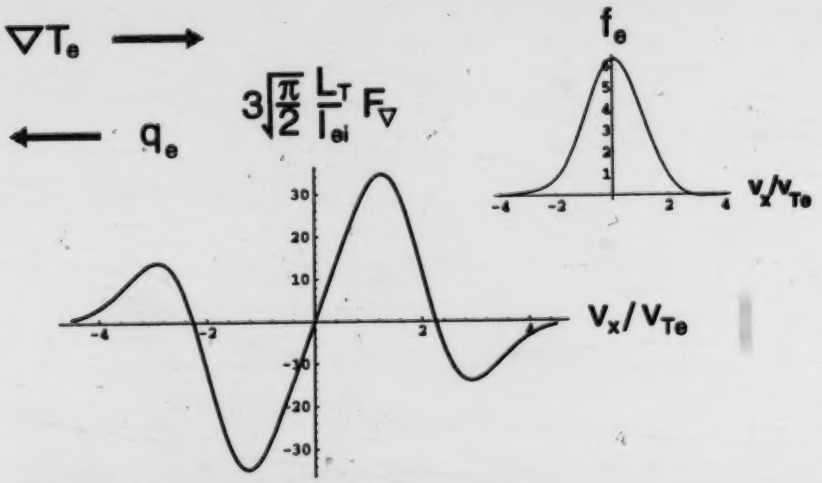

Figure 6.2.1: Plot of the electron distribution function for the background state f_e, and the anisotropic correction $3\sqrt{\pi/2}\,L_T/l_{ei}\;F_\nabla$ as a function of v_x/v_{Te}.

for the background state. In this figure the temperature gradient is to the right, and the heat flux to the left. The fast heat carrying electrons can clearly be seen in the tail of the distribution function f_e for velocities $v_x \sim -3.5\,v_{Te}$. To retain neutrality of the plasma, slower electrons must flow in the opposite direction. This return current is responsible for skewing the distribution function at small velocities $v \sim c_s/v_{Te} \sim 10^{-2}$. In the collisionless regime this effect is responsible for the reduction of the ion acoustic damping rate for waves propagating in the direction of the temperature gradient.

As a check one may calculate the heat flux $\vec{q}_e$ from the background state (6.8).

The result is

$$q_e = -\frac{128}{3\pi} n_e T_e v_{Te} \left(\frac{l_{ei}}{L_T}\right),$$
$$= -\frac{128}{3\pi} n_e v_{Te} l_{ei} \frac{dT}{dx}, \tag{6.9}$$

which corresponds exactly to the Braginskii result for $Z \gg 1$ (4.55). Equations (6.7) and (6.8) provide the full definition of the background state about which we will calculate the dynamic form factor $S(k,\omega)$.

6.3 Nonlocal closure

We use the method of Bychenkov *et al.* [8] described in Chapter 4 to reduce the kinetic equation (4.32), now linearized around the background state (6.5, 6.8) from Section 6.2, into a set of hydrodynamical equations with nonlocal closure relations that will not be restricted by the conditions of strong collisions. The general method remains exactly the same as in Section 4.4. We assume slow plasma motions, large ionization Z, and account for electron-electron collisions only in the symmetric part of the distribution function. In addition l_{ei}/L_T, coming from the background state is assumed to be a small parameter, for the validity of the analysis in Section 6.2.

Expanding the phase space fluctuation in Legendre polynomials $\delta f_e(v,\mu) = \sum_{l=0}^{\infty} P_l(\mu)\delta f_l(v)$, where $\mu = \vec{k}\cdot\vec{v}/kv$, the infinite set of equations for the angular harmonics becomes

$$\frac{\partial \delta f_0}{\partial t} + \frac{i}{3}kv\delta f_1 - \frac{i}{3}kv\delta u_i \frac{\partial F_0}{\partial \vec{v}}$$
$$+\frac{i}{3}\frac{e}{m_e}k\delta\phi\left(\frac{\partial F_\nabla}{\partial v} + 2\frac{F_\nabla}{v}\right) = C_{ee}[\delta f_0], \tag{6.10}$$

$$ikv\delta f_0 + i\frac{e}{m_e}k\delta\phi\frac{\partial F_0}{\partial v}$$
$$-ikv(\tilde{\nu}_1 - \nu_{ei})\left[-i\frac{\delta u_i}{kv}\frac{\partial F_0}{\partial v} + i\frac{e}{m_e}\frac{\delta\phi}{kv^2}\left(\frac{\partial F_\nabla}{\partial v} - \frac{F_\nabla}{v}\right)\right] = -\tilde{\nu}_1\delta f_1. \tag{6.11}$$

The renormalized collision frequency $\tilde{\nu}_1$ (4.87) has been used to include the effects of all the higher angular harmonics, necessary for the description of Landau damping.

The solution to the linear equation for δf_0 obtained by eliminating δf_1 between the equations (6.10, 6:11) can be expressed as a linear combination of four basis functions ψ^A $(A = N, R, T, \nabla)$, after a Laplace transform in time (Laplace variable p), and using the initial conditions (4.91),

$$\delta f_0 = \frac{e\delta\phi}{T_0} F_0 + \sum_A P_A \psi^A F_0. \tag{6.12}$$

The definitions of $P_N = \delta n_e(0)/n_{0e} - pe\delta\phi/T_{0e}$, $P_T = 3\delta T_e(0)/2T_{0e}$, and $P_R = -ik\delta u_i$ are the same as in Section 4.4.2 of Chapter 4, but in addition the presence of the background heat flux introduces the new term $P_\nabla = -ikv_{Te}(l_{ei}/L_T)(e\delta\phi/T_e)$ corresponding to the extra basis function ψ^∇. The ψ^A in (6.12) must satisfy the equations

$$\left(\frac{k^2v^2}{3\tilde{\nu}_1} + p\right)\psi^A = F_0^{-1} C_{ee}[F_0\psi^A] + S_A, \tag{6.13}$$

with three different source terms $S_N = 1$, $S_T = v^2/3v_{Te}^2 - 1$, $S_R = v^2\nu_{ei}/3v_{Te}^2\tilde{\nu}_1$ present in the original theory [8], and a new source term in the extra equation for ψ^∇,

$$\begin{aligned} S_\nabla &= \frac{1}{F_0}\frac{v_{Te}}{v}\left[F_H + \frac{v^2}{3H_1}\frac{\partial}{\partial v}\left(\frac{F_H}{v}\right)\right]\frac{L_T}{l_{ei}}, \\ &= \frac{4}{3\sqrt{\pi}}\left[x^{3/2}(4-x) - \frac{2}{3H_1}x^{3/2}\left(6 - \frac{13}{2}x + x^2\right)\right], \end{aligned} \tag{6.14}$$

where we have introduced the dimensionless energy variable $x = v^2/2v_{Te}^2$. After calculating the moments $\delta n_e(t)$ and $\delta T_e(t)$ from (6.12), and eliminating the initial conditions P_N and P_T the result for δf_1 is

$$\begin{aligned} \delta f_1 = &-\frac{e\delta E^* v}{T_0\tilde{\nu}_1}\frac{J_T^T\psi^N - J_T^N\psi^T}{D_{NT}^{NT}}F_0 - i\frac{kv}{\tilde{\nu}_1}\frac{\delta T}{T_0}\frac{(J_N^N + J_N^T)\psi^T - (J_T^T + J_T^N)\psi^N}{D_{NT}^{NT}}F_0 \\ &-\frac{v\delta u_i}{\tilde{\nu}_1 v_{Te}^2}\left[\tilde{\nu}_1 - \nu_{ei} + k^2v_{Te}^2\left(\psi^R + \frac{D_{NT}^{TR}}{D_{NT}^{NT}}\psi^N - \frac{D_{NT}^{NR}}{D_{NT}^{NT}}\psi^T\right)\right]F_0 \\ &-v_{Te}\frac{l_{ei}}{L_T}\frac{k^2v}{\tilde{\nu}_1}\left(\psi^\nabla - \frac{D_{NT}^{\nabla T}}{D_{NT}^{NT}}\psi^N - \frac{D_{NT}^{N\nabla}}{D_{NT}^{NT}}\psi^T\right)F_0 - \frac{\tilde{\nu}_1 - \nu_{ei}}{\tilde{\nu}_1}v_{Te}^2\frac{\partial}{\partial v}\left(\frac{F_\nabla}{v}\right)\frac{e\delta\phi}{T_e}. \end{aligned} \tag{6.15}$$

The J_B^A and D_{AB}^{CD} moments (4.97) have been extended to include the moments of ψ^∇, $(A = \nabla)$ also,

$$J_B^\nabla = \frac{4\pi}{n_e}\int_0^\infty v^2 dv \psi^\nabla F_0 S_B. \tag{6.16}$$

The solution for δf_1 (6.15) is used to construct the closure relations by calculating the current $\delta\vec{j}$ (4.99), heat flux $\delta\vec{q}_e$ (4.101) and friction $\delta\vec{R}_{ie}$ (4.103)

$$\delta\vec{j} = \sigma\,\delta\vec{E}^* + \alpha\, i\vec{k}\delta T_e + \beta_j e n_e\,\delta\vec{u}_i - p_\nabla\,\beta_j^\nabla e n_e c_s\left(\frac{e\delta\phi}{T_e}\right), \tag{6.17}$$

$$\delta\vec{q}_e = -\alpha T_e\,\delta\vec{E}^* - \chi\, i\vec{k}\delta T_e - \beta_q n_e T_e\,\delta\vec{u}_i - p_\nabla\,\beta_q^\nabla n_e T_e c_s\left(\frac{e\delta\phi}{T_e}\right), \tag{6.18}$$

$$\begin{aligned}\delta\vec{R}_{ie} &= -(1-\beta_j)n_e e\,\delta\vec{E}^* + \beta_q n_e\, i\vec{k}\delta T_e \\ &\quad -\beta_r m_e n_e v_{Te}/l_{ei}\,\delta\vec{u}_i + p_\nabla\,\beta_r^\nabla m_e n_e c_s^2/l_{ei}\left(\frac{e\delta\phi}{T_e}\right).\end{aligned} \tag{6.19}$$

The definition $p_\nabla = v_{Te}l_{ei}/c_s L_T$ has been made that is a measure of the heat flux in the background state. In addition to the transport coefficients σ, α, χ, β_j, β_q, and β_r from Ref. [8] there are the new coefficients

$$\beta_j^\nabla = \left(\frac{D_{NT}^{\nabla T}}{D_{NT}^{NT}}\right), \qquad \beta_q^\nabla = \left(\frac{D_{NT}^{\nabla T} + D_{NT}^{\nabla N}}{D_{NT}^{NT}}\right), \tag{6.20}$$

$$\begin{aligned}\beta_r^\nabla = -\Bigg[k^2 v_{Te}l_{ei}\left(J_R^\nabla - \frac{D_{NT}^{\nabla T}}{D_{NT}^{NT}}J_R^N - \frac{D_{NT}^{N\nabla}}{D_{NT}^{NT}}J_R^T\right) \\ -\frac{4}{3\sqrt{\pi}}\int_0^\infty dx \frac{\sqrt{x}\exp(-x)}{H_1}(6 - \frac{13}{2}x + x^2)\Bigg],\end{aligned} \tag{6.21}$$

that are a result of the heat carrying background state. The electron temperature moment of (6.10), the ion equations (5.6-5.8) and the closure relations (6.17-6.19), (6.20, 6.21) provide a closed set of equations suitable for describing the evolution of the hydrodynamic fluctuations n_a, $\vec{u}_a$, and T_a for arbitrary ratio of fluctuation wavenumber to electron-ion and ion-ion mean free path around the background state (6.5, 6.8).

6.3.1 Current free form of the transport equations

For slow motions, the plasma is quasineutral $\delta n_e \sim Z\delta n_i$, and current free $\delta j = 0$. Setting the current equal to zero $\delta j = 0$ in (6.17) allows us to solve for the potential $\delta\phi$, which to first order in the expansion parameter l_{ei}/L_T gives

$$\frac{e\delta\phi}{T_e} = \frac{\delta n}{n_e} + \left(1+\frac{e\alpha}{\sigma}\right)\frac{\delta T_e}{T_e} - i\beta_j\frac{e^2 n_e}{\sigma T_e}\delta u_i$$
$$+ip_\nabla \frac{e^2 n_e c_s}{k\sigma T_e}\beta_j^\nabla\left[\frac{\delta n}{n_e} + \left(1+\frac{e\alpha}{\sigma}\right)\frac{\delta T_e}{T_e} - i\beta_j\frac{e^2 n_e}{\sigma T_e}\delta u_i\right]. \qquad (6.22)$$

This expression for the potential (6.22) can now be used in order to eliminate $\delta\phi$ from the expressions for heat flux (6.18) and friction (6.19) and obtain the current free forms

$$\delta\vec{q}_e = -\kappa i\vec{k}\,\delta T_e - \beta n_e T_e\,\delta u_i$$
$$-p_\nabla\,\beta^\nabla n_e T_e c_s\frac{\delta n_e}{n_e} - p_\nabla\,\beta^\nabla n_e T_e c_s\left(1+\frac{e\alpha}{\sigma}\right)\frac{\delta T_e}{T_e} + ip_\nabla\,\beta^\nabla n_e T_e c_s\beta_j\frac{e^2 n_e}{k\sigma T_e}\delta\vec{u}_i, \qquad (6.23)$$

$$\delta\vec{R}_{ie} = n_e\left(\beta+\frac{e\alpha}{\sigma}\right)i\vec{k}\delta T_e + (1-\beta_j)\beta_j\frac{e^2 n_e^2}{\sigma}\delta\vec{u}_i - \beta_r\frac{m_e n_e v_{Te}}{l_{ei}}\delta\vec{u}_i$$
$$-p_\nabla\left((1-\beta_j)\beta_j^\nabla\frac{e^2 n^2}{\sigma}c_s - \beta_r^\nabla\frac{m_e n_e v_{Te}}{l_{ei}}c_s\right)\left[\frac{\delta n_e}{n_e} + \left(1+\frac{e\alpha}{\sigma}\right)\frac{\delta T_e}{T_e} - i\beta_j\frac{e^2 n_e}{\sigma T_e}\delta u_i\right] \qquad (6.24)$$

The first two terms of (6.23) are the same as in Ref. [8], but there are in addition new terms proportional to p_∇ that come from the heat flux in the background state. One of the effects of these new terms is to couple the fluctuations in the heat flux δq_e to the fluctuations in density δn_e (6.23). In previous work [64] such a coupling has been considered due to the density dependence of the electron thermal conductivity

$$\delta q_e = -\kappa_e\, ik\delta T_e - \delta n_e\frac{\partial\kappa_e}{\partial n_e}\nabla T_e - \delta T_e\frac{\partial\kappa_e}{\partial T_e}\nabla T_e. \qquad (6.25)$$

Classical thermal conductivity (4.55) is not density dependent $\partial\kappa_e/\partial n_e = 0$, but in the weakly collisional regime defined by $kl_{ei} \gtrsim 0.01$, the nonlocal expressions [50, 51, 52] contain a density dependence through the electron-ion mean free path kl_{ei}. Such a coupling has been shown to destabilize ion acoustic waves by changing the sign of the

heat flux, causing more heat to flow into hotter regions [64]. Such a mechanism can come into play in our equations (6.23).

6.3.2 Hydrodynamic moment equations

The expression for the potential $\delta\phi$ (6.22), the heat flux $\vec{q}_e$ (6.23), and friction $\delta\vec{R}_{ie}$ (6.24) may be used to close the hydrodynamic moment equations for the fluctuating quantities. The equation for electron density δn_e is obtained from (6.10) by taking the velocity moment $4\pi \int_0^\infty dv\ v^2$, and the electron temperature fluctuation the moment $4\pi m_e/3n_e \int_0^\infty dv\ v^2(v^2 - 3v_{Te}^2)$. The density moment gives the electron continuity equation

$$\frac{\partial}{\partial t}\delta n_e + n_e i\vec{k}\cdot\delta\vec{u} = 0, \tag{6.26}$$

and the temperature moment gives

$$\frac{\partial}{\partial t}\delta T_e + \frac{2}{3n_e}i\vec{k}\cdot\delta\vec{q}_e + \frac{2}{3}T_e i\vec{k}\cdot\delta\vec{u} = 0, \tag{6.27}$$

which is the same as in Chapter 5. There is no heating term in (6.27) $2/3n_e\ \delta\vec{E}\cdot\vec{j}$ due to the lack of current flow in the background state, $\vec{j} = 0$. These moment (6.26, 6.27) equations are supplemented with the ion equations from Section 4.6.2

$$\begin{aligned}\frac{\partial}{\partial t}\delta\vec{u} &= -\frac{Ze}{m_i}i\vec{k}\delta\phi - \frac{i\vec{k}}{m_i n_i}(n_i\delta T_i + T_i\delta n_i) \\ &\quad + \frac{1}{m_i n_i}i\vec{k}\cdot\delta\hat{\sigma}_i + \frac{1}{m_i n_i}\delta\vec{R}_{ie}, \qquad (6.28)\\ \frac{\partial}{\partial t}\delta T_i &+ \frac{2}{3}T_i i\vec{k}\cdot\delta\vec{u} = 0, \qquad (6.29)\end{aligned}$$

which will allow the calculation of the correlations of any of the hydrodynamic variables by the method of Chapter 5.

6.3.3 Calculation of the dynamic form factor

The calculation of the dynamic form factor $S(k,\omega)$ parallels exactly that of Section 5.2.4, but now using the modified closure relations for the potential, heat flux and friction to the set of moment equations (6.26-6.29). After taking a time Laplace transform, the density fluctuation $\delta n_e(k,\omega)$ is solved for in terms of its initial value

$$D(k,\omega)\delta n_e(k,\omega) = \delta n_e(0), \tag{6.30}$$

where the dispersion relation $D(k,\omega)$ becomes

$$D(k,\omega) = -i\omega + i\,\frac{k^2 c_s^2(1 + p_\nabla\, ia) + k^2 v_{Ti}^2}{\omega - \Delta + 2i\gamma_a}. \tag{6.31}$$

In (6.31) the γ_a (5.24), Δ (5.26), and v_s (5.28) have already been defined, the effects of the background state on the dispersion are ignored, and a is given by

$$a = kc_s\left[(1-\beta)\beta^\nabla\frac{n_e}{k^2\kappa} + \beta_j\beta_j^\nabla\frac{1}{k^2}\frac{en_e}{\sigma T} + \frac{1}{k^2}\frac{\beta_r^\nabla}{v_{Te}l_{ei}}\right]. \tag{6.32}$$

Multiplying (6.30) by $\delta n_e^*(0)$ and averaging gives ,

$$\langle\delta n_e(k,\omega)\delta n_e(-k,0)\rangle = \frac{\langle\mid \delta n_e(k)\mid^2\rangle}{D(k,\omega)}. \tag{6.33}$$

Using the relation $S(k,\omega) = 2\mathrm{Re}\langle\delta n_e(k,\omega)\delta n_e(-k,0)\rangle/n_e$ and assuming the plasma is weakly coupled $\langle\mid \delta n_e(k)\mid^2\rangle = n_e$ gives,

$$S(k,\omega) = 2\,\mathrm{Re}\left(\frac{1}{D(k,\omega)}\right), \tag{6.34}$$

with the result,

$$S(k,\omega) = 2\frac{2k^2(c_s^2 + v_{Ti}^2)\gamma_a - p_\nabla\,\omega_s^2\omega a}{(\omega^2 - k^2v_s^2)^2 + (2\omega\gamma_a - p_\nabla\,\omega_s^2 a)^2}. \tag{6.35}$$

6.4 Analysis of the results

The expression for the dynamic form factor (6.35) takes into account the effect of a heat flux on the electron density fluctuation spectrum. The heat flux q_e is parameterized through p_∇, by $q_e = 128/3\pi\, n_eT_ec_s\,p_\nabla$. For zero heat flux ($p_\nabla = 0$), the

expression (6.35) reduces to that found in Chapter 4 (5.27). As expected the peak heights are asymmetric. This can clearly be seen in Figure 6.4.1 where a comparison is made between the expression (6.35), both with and without a heat flux. The ion acoustic fluctuations in the direction of the temperature gradient are enhanced compared to ion acoustic fluctuations traveling in the opposite direction. Figure 6.4.2 shows a comparison between the predictions of our nonlocal formulation (6.35) and collisionless Vlasov theory (2.18), in the weakly collisional regime $kl_{ei} \sim 1$. The peak height asymmetry is much reduced from that of the collisionless predictions. This can be understood, because for $kl_{ei} \sim 1$ ion acoustic damping γ_a is larger than Landau damping would predict cf. eg. Fig. 5.3.1. Figure 6.4.3 shows the spectrum for a collisionless plasma. The nonlocal expression (6.35) and collisionless theory (2.18) are in close agreement. In fact, for $kl_{ei} \gg 1$, the threshold for instability

$$p_\nabla = \frac{2\omega\gamma_a}{\omega_s^2 a} \tag{6.36}$$

agrees with the collisionless result $p_\nabla \to 2/3$. The collisionless result can be obtained on substitution of the distribution function (6.5, 6.8) into the collisionless expression for $S(k,\omega)$ (2.18).

The peak height asymmetry is significantly different for nearly collisionless electrons kl_{ei} and weakly collisional electrons $kl_{ei} \sim 1$ in comparison with Vlasov theory. This suggests the effect may be measurable in an experiment, since peak height asymmetry is easier to measure than line widths. One could consider an experiment where the scattering angle is changed from nearly forwards, say 10°, to the backwards direction 180°. In this way the collisionality of the fluctuations probed could be chosen to vary from the weakly collisional to the collisionless, and the asymmetries compared with both theories.

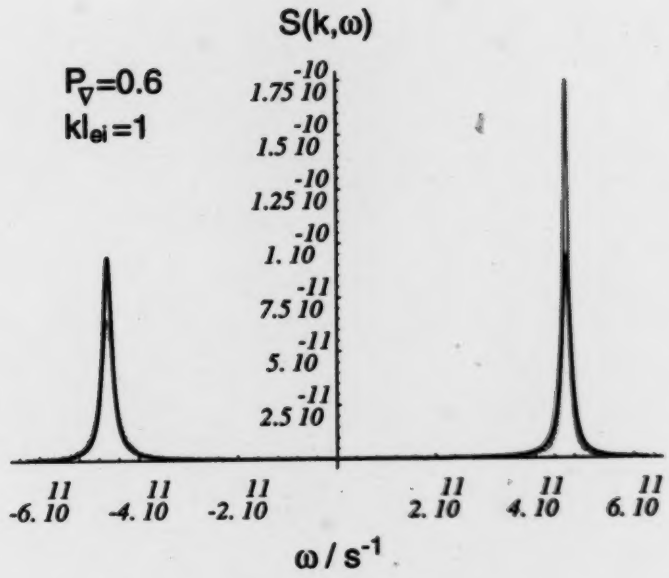

Figure 6.4.1: Nonlocal model without gradient (black lines) compared to the case with gradient

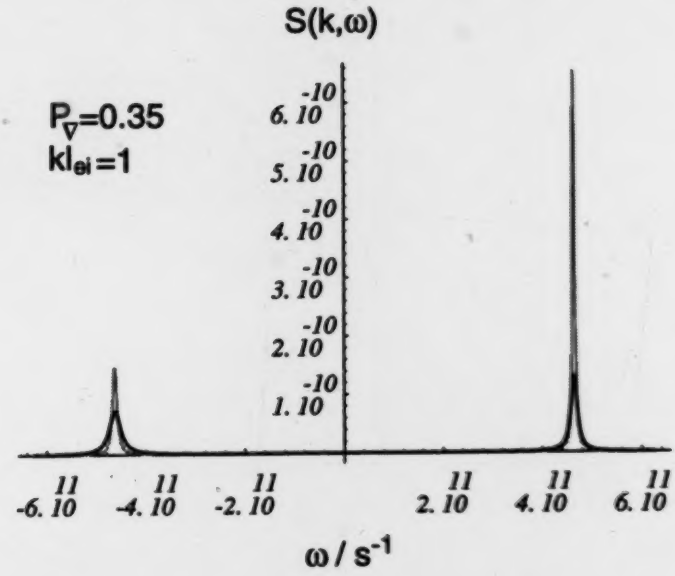

Figure 6.4.2: Comparison of the nonlocal model with collisionless theory in the weakly collisional regime.

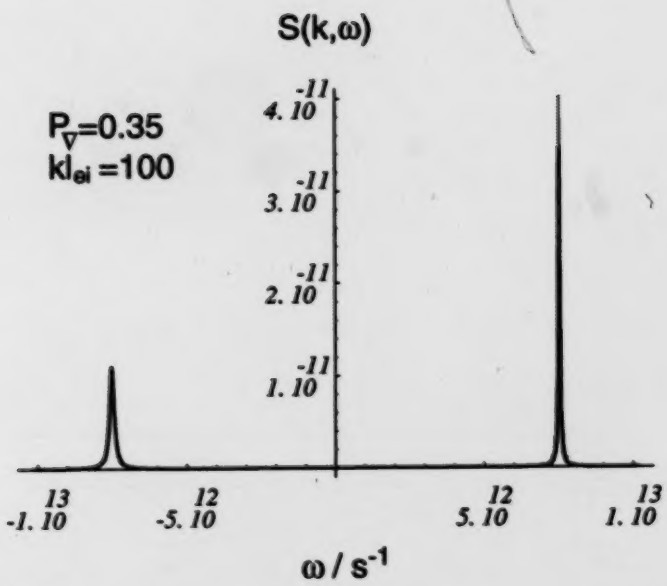

Figure 6.4.3: This is the comparison of the nonlocal model (black lines) with the collisionless theory (solid lines).

Chapter 7

CONCLUSION

The importance and range of applicability of Thomson scattering as a plasma diagnostic technique depends on the accuracy of the theoretical model of fluctuations and scattering cross-section. We have described a theory for the dynamical form factor $S(k,\omega)$, which is valid for arbitrary particle collisionality in plasmas with large Z and ZT_e/T_i. Our theory properly describes the ion acoustic resonance in the entire region of parameters between collision dominated hydrodynamics and the collisionless formulation based on the Vlasov description. This has been achieved using generalized nonlocal hydrodynamics [64, 8, 74] for the fluctuating variables.

The starting point has been an exact result of fluctuation theory [6] which demonstrates that the two point correlation function of the phase space fluctuation satisfies the usual linearized kinetic equation with the Landau collision operator. We have solved this equation and reduced the problem of finding fluctuations of the phase space densities to the solution of the linear generalized hydrodynamical equations for the fluctuating hydrodynamical variables. The closure leading to the hydrodynamical model has been achieved with the help of frequency dependent ion transport coefficients [64] and the full set of nonlocal electron transport coefficients [8]. This derivation involves the frequency dependent Grad 21-moment approximation for the ion fluctuations and a generalized Laguerre expansion of the electron fluctuation density. Calculations of the dynamical form factor, $S(k,\omega)$, are completed assuming an equilibrium electron density correlation function at the initial moment in time.

Starting from our general theory of the dynamic form factor, we have analyzed

in detail two different regimes of ion acoustic fluctuations with weakly collisional electrons and cold ions, $kl_i \ll 1$, $kl_{ei} \sim 1$, and with weakly collisional ions and collisionless electrons, $kl_i \sim 1$, $kl_{ei} \gg 1$. Equation (5.27) provides an expression for the dynamical form factor in the first limit, of weakly collisional electrons. The k-dependent transport coefficients are calculated by a Fortran code `nonlocal.f` that is described in Appendix C.1. The ion acoustic resonance line shape calculated from (5.27) has been used to demonstrate the effect of nonlocal inhibited electron thermal transport. The possibility of directly inferring electron thermal transport properties from Thomson scattering measurements is proposed for realistic experimental parameters. Equations (5.27) and (5.30) give an expression for $S(k,\omega)$ in the regime of weak ion collisionality and for collisionless electrons. This is the regime of parameters often encountered in X-ray lasers plasmas [35, 36], where our theory predicts variations of the Thomson scattering cross-section which are consistent with experimental observations.

The derivation of the dynamic form factor described above has been repeated for a plasma carrying a heat flux supported by an external source. The two point correlation function of the phase space fluctuation then satisfies the Landau kinetic equation linearized around a heat carrying background state. The background state is characterized by a temperature gradient that is sufficiently long, so that the classical collisional description can be used to describe it. The closure of the kinetic equation leading to the nonlocal hydrodynamical equations proceeds as before, but three new k-dependent coefficients are introduced as a result of the background heat flux. These new coefficients may be calculated by the Fortran code `gradient.f` which is listed in Appendix C.1.

The dynamic form factor can be calculated from the resulting nonlocal hydrodynamic equations and leads to the expression (6.35). This expression reduces to our previous expression (5.27) for zero heat flux, but develops an asymmetry in the ion acoustic peaks when a heat flux is present. This asymmetry is in agreement with

collisionless Vlasov theory in the collisionless limit, but shows drastic differences in the weakly collisional regime. Such a difference should be observable in a Thomson scattering experiment.

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Appendix A

A.1 Derivation of the Thomson scattering cross section

It is particularly useful when considering radiation from moving charges to employ the Liénard-Wiechert potentials, as described in chapter 14 of the book by Jackson [76]. In terms of these

$$\vec{E}(\vec{r},t) = e\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})R^2}\bigg|_{ret} + \frac{e}{c}\,\frac{\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{(1-\hat{n}\cdot\vec{\beta})^3 R}\bigg|_{ret}, \tag{A.1}$$

$$\vec{B}(\vec{r},t) = \hat{n}\times\vec{E}(\vec{r},t)|_{ret}, \tag{A.2}$$

where $\vec{R}$ is the displacement between the observer at the position $\vec{r}$ and the radiating particle of charge e at the position $\vec{\rho}$, $\vec{R}=\vec{r}-\vec{\rho}$. The unit vector in the direction of $\vec{R}$ is denoted by $\hat{n}$. $\vec{\beta}$ is the velocity of the charge $\vec{v}$ divided by the speed of light c, $\vec{\beta}=\vec{v}/c$ and a dot denotes a time derivative. $\gamma=\sqrt{1+\beta^2}$. The subscript "ret" implies that all quantities are to be evaluated at the retarded time, $t'=t-(R(t')/c)$.

At a point $\vec{r}$ in the wave zone far away from the charge $R \gg 1$, the first electrostatic term will be negligible, leaving

$$\vec{E}(\vec{r},t) \approx \frac{e}{c}\,\frac{\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{(1-\hat{n}\cdot\vec{\beta})^3 R}\bigg|_{ret}. \tag{A.3}$$

This simplifies if the charge is nonrelativistic $\beta \ll 1$,

$$\vec{E}(\vec{r},t) \approx \frac{e}{c^2}\,\frac{\hat{n}\times(\hat{n}\times\vec{a})}{R}\bigg|_{ret}. \tag{A.4}$$

where $\vec{a}$ is the acceleration of the charge. Corresponding to the Thomson scattering probe we consider a plane monochromatic electromagnetic wave is incident upon the plasma,

$$\vec{E}_i(\vec{x},t) = \mathrm{Re}\{\vec{E}_0 \exp i(\vec{k}_0 \cdot \vec{x} - \omega_0 t)\}, \tag{A.5}$$

with the wavenumber $\vec{k}_0$ and frequency ω_0. All the charges in the plasma will oscillate in this field with the acceleration of particles given by the real part of

$$\vec{a}(\vec{\rho},t) = \frac{e}{m}\vec{E}_0 \exp i(\vec{k}_0 \cdot \vec{\rho} - \omega_0 t). \tag{A.6}$$

The electrons will undergo the largest accelerations due to the smallness of their mass, $m_e \ll m_i$. Hence, to a very good approximation only the electrons will contribute to the scattered light. Further simplifications will be that the scattered signal is weak and most of the light is transmitted. Under this approximation each particle will see the same field (Born approximation). The acceleration experienced by an electron in the $\vec{E}$ field can then be easily solved for (assuming $v \ll c$). The Electric field emitted by each electron is obtained by substituting expression (A.6) into (A.4). The scattered field $\vec{E}_s$ at the observation point $\vec{r}$ will be the vector sum of the fields from all the different electrons,

$$\vec{E}_s(\vec{r},t) = \frac{e^2}{mc^2}\int dt' \int d\vec{\rho} \sum_j \delta[\vec{\rho} - \vec{\rho}_j(t')]\,\hat{n} \times (\hat{n} \times \vec{E}_0)$$
$$\frac{\exp i(\vec{k} \cdot \vec{\rho}(t') - \omega_0 t')}{\mid \vec{r} - \vec{\rho}(t') \mid}\delta(t' - t + \mid \vec{r} - \vec{\rho}(t') \mid /c). \tag{A.7}$$

Since we are in the wave zone, we may approximate $R(t')$ by $R(t') \approx R - \hat{n} \cdot \vec{\rho}(t')$, and we also make use of the vector identity $\hat{n} \times (\hat{n} \times \vec{E}_0) = \hat{n}(\hat{n} \cdot \vec{E}_0) - \vec{E}_0$. On taking the representation of the delta function

$$\delta(t' - t + \mid \vec{r} - \vec{\rho}(t') \mid /c) = \int \frac{d\omega}{2\pi} \exp i\omega\,(t' - t + \mid \vec{r} - \vec{\rho}(t') \mid /c), \tag{A.8}$$

this results in an expression for the electric field $\vec{E}_s(\vec{r},t)$ at the point $\vec{r}$ at the time t,

$$\vec{E}_s(\vec{r},t) = -\frac{r_0}{R}[\vec{E}_0 - \hat{n}(\hat{n} \cdot \vec{E}_0)]\mathrm{Re}\int \frac{d\omega_s}{2\pi} \exp -i\omega_s(t - R/c)$$

$$\int d\vec{\rho} \int dt' \exp -i(\omega_0 - \omega_s)t' \exp\{i(\vec{k}_0 - \frac{\omega_s}{c}\hat{n}) \cdot \vec{\rho}\} \sum_j \delta(\vec{\rho} - \vec{\rho}_j(t')). \quad \text{(A.9)}$$

We actually need to evaluate the average power of scattered light in the solid angle $d\Omega$ centered around $\vec{r}$

$$P_s(\vec{r})d\Omega = \frac{cR^2}{4\pi} d\Omega \lim_{T\to\infty} \frac{1}{T} \int \langle \mid \vec{E}_s \mid^2 \rangle. \quad \text{(A.10)}$$

Using

$$\begin{aligned} \mid \vec{E}_0 - \hat{n}(\hat{n} \cdot \vec{E}_0) \mid^2 = & \\ & \mid \vec{E}_0 \mid^2 - (\hat{n} \cdot \vec{E}_0)^2 \\ & = \mid \vec{E}_0 \mid (1 - \sin^2\theta \cos^2 \mid \phi_0 - \phi \mid), \end{aligned} \quad \text{(A.11)}$$

where θ is the angle between $\vec{k}_0$, the wavevector of the incident light, and $\vec{r}$, and the azimuthal angles ϕ_0 and ϕ locate the position of the electric field vector $\vec{E}_0$ and $\vec{r}$ in the plane perpendicular to $\vec{k}_0$, the result is

$$P_s(\vec{r}, \omega_s)d\Omega d\omega_s = \frac{P_i r_0^2}{2\pi A} d\Omega d\omega_s (1 - \sin^2\theta \cos^2 \mid \phi_0 - \phi \mid) N_e S(\vec{k}, \omega) \quad \text{(A.12)}$$

where

$$S(\vec{k}, \omega) = \lim_{T\to\infty, V\to\infty} \frac{1}{TV} \langle \frac{|n_e(\vec{k}, \omega|^2}{n_e} \rangle, \quad \text{(A.13)}$$

and P_i is the incident power, $P_i = (cE_0^2/8\pi)A$, for a beam of cross-sectional area A.

Appendix B

B.1 Plasma dispersion function

The plasma dispersion function is defined by,

$$\epsilon(\vec{k},\omega) = 1 + \sum_{\alpha} \frac{{\omega_{p\alpha}}^2}{k^2} \int d^3v \frac{\vec{k} \cdot \partial f^{\alpha}(\vec{v})/\partial \vec{v}}{\omega - \vec{k} \cdot \vec{v} + i\epsilon}. \tag{B.1}$$

Where we will write

$$\epsilon(\vec{k},\omega) = 1 + \sum_{\alpha} \chi_{\alpha}. \tag{B.2}$$

Here the χ_{α} are the susceptibilities of particles of the species α. Now we would like to evaluate this for the case of a Maxwellian plasma,

$$f_M^{\alpha}(v) = \frac{1}{(2\pi)^{3/2} {v_{T\alpha}}^3} \exp\left(-\frac{v^2}{2{v_{T\alpha}}^2}\right). \tag{B.3}$$

The result for an electron-ion plasma is

$$\chi_e = \alpha^2 \left[Rw(x_e) + iIw(x_e)\right] \quad \text{and} \quad \chi_i = \alpha^2 \frac{ZT_e}{T_i} \left[Rw(x_i) + iIw(x_i)\right]. \tag{B.4}$$

Where

$$x_{\alpha} = \frac{1}{\sqrt{2}} \frac{\omega}{k v_{T\alpha}}, \quad v_{T\alpha} = \sqrt{\frac{T_{\alpha}}{m_{\alpha}}} \text{and} \quad \alpha = \frac{1}{k\lambda_{De}}. \tag{B.5}$$

$Rw(x)$ and $Iw(x)$ are the real and imaginary parts of the plasma dispersion function tabulated by Fried and Conte [82]. With these definitions the modulus squared of the plasma dielectric function is given by

$$\begin{aligned} |\epsilon|^2 = &\left[1 + \alpha^2 \left(Rw(x_e) + \frac{ZT_e}{T_i} Rw(x_i)\right)\right]^2 \\ &+ \left[\alpha^2 Iw(x_e) + \alpha^2 \frac{ZT_e}{T_i} Iw(x_i)\right]^2 \end{aligned} \tag{B.6}$$

For real x we have

$$Rw(x) = 1 - 2x\,exp(-x^2)\int_0^x dp\ \exp(p^2), \tag{B.7}$$

$$Iw(x) = \sqrt{\pi}x\ \exp(-x^2). \tag{B.8}$$

The error function is defined by

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z dy\ \exp(-y^2). \tag{B.9}$$

We can write the integral in Eq. (B.7) in terms of this

$$\int_0^x dy\ \exp(y^2) = -i\frac{\sqrt{\pi}}{2}erf(ix). \tag{B.10}$$

This allows us to use Mathematica's intrinsic function `Erfi(z)`, [78]

$$\texttt{Erfi}\,(z) = -i\,erf\,(iz). \tag{B.11}$$

B.2 Dynamic form factor in collisionless plasma

We now have all that is required for $S(k,\omega)$. In terms of the plasma response functions, it looks like

$$S(k,\omega) = \frac{2\pi}{k}\frac{|\,1+\chi_i\,|^2 f_{e0}(\omega/k) + Z|\,\chi_e\,|^2 f_{i0}(\omega/k)}{|\,\epsilon\,(k,\omega)\,|^2}. \tag{B.12}$$

Where here the $f_{a0}(\omega/k)$ are the one dimensional distribution functions evaluated at the phase velocity. On substitution of our explicit forms for the partial susceptibilities and the dielectric function, this becomes

$$S(k,\omega) = \frac{\sqrt{2\pi}}{kv_{Te}}\left[\frac{A_e}{|\epsilon|^2} + \frac{A_i}{|\epsilon|^2}\right], \tag{B.13}$$

where

$$A_e = e^{-x_e^2}\left[\left(1+\alpha^2\frac{ZT_e}{T_i}Rw(x_i)\right)^2 + \left(\alpha^2\frac{ZT_e}{T_i}Iw(x_i)\right)^2\right], \tag{B.14}$$

and

$$A_i = Z\left[\left(\frac{ZT_e}{T_i}\right)\left(\frac{m_i}{Zm_e}\right)\right]^{\frac{1}{2}} e^{-x_i^2}\left[\left(\alpha^2 Rw(x_e)\right)^2 + \left(\alpha^2 Iw(x_e)\right)^2\right]. \tag{B.15}$$

Appendix C

C.1 Numerical evaluation of transport coefficients

A Fortran 77 code "`nonlocal.f`" has been written, that efficiently calculates the ionization Z, and collisionality kl_{ei} dependent transport coefficients $\sigma(Z, kl_{ei})$, $\chi(Z, kl_{ei})$, $\alpha(Z, kl_{ei})$, $\beta_j(Z, kl_{ei})$, $\beta_q(Z, kl_{ei})$, and $\beta_r(Z, kl_{ei})$ that enter in the nonlocal theory of electron transport due to [8] described in Chapter 4. In the original paper [8] no tables of coefficients were provided. The workings of the code are explained in some detail in this Appendix, The code will be useful for anyone wishing to make use of our expressions for the dynamic form factor (5.27) described in Chapter 5. The basic model can be extended, in that it is relatively easy to include extra source terms into the equations (4.93), coming from non-Maxwellian background states or external sources.

In Section C.2 the code "`gradient.f`" is described which is an extension of `nonlocal.f` to calculate the new transport coefficients $\beta_j^{\nabla}(Z, kl_{ei})$, $\beta_q^{\nabla}(Z, kl_{ei})$, and $\beta_r^{\nabla}(Z, kl_{ei})$, that enter in the theory of Chapter 6 due to the presence of a heat carrying background state. Other examples that have been discussed include the effect of inverse bremsstrahlung heating and the ponderomotive force [80]. For reference, a listing of the code `gradient.f` is given in Section C.3. The subroutines `gaulag`, `gami3h`, `choldc`, and `cholsl` are to be taken from *Numerical Recipes* [79].

C.1.1 Operation of the code nonlocal.f

The code requires as input, the values for the variables, **npoly**, $\mathbf{kl_{ei}}$ and Z. These are the number of polynomials to use in the truncation of the expansion (4.110) for the functions ψ^A

$$\psi^A = \frac{Zkl_{ei}}{kv_{Te}} \sum_{n=0}^{\infty} c_n^A L_n^{(1/2)}(x), \tag{C.1}$$

the particle collisionality, and the ionization respectively. The maximum size for **npoly** is set by the variable **nphys**, which is the physical dimension of the arrays defined in the program. The first step is to evaluate the matrix elements of A (4.113)

$$\begin{aligned} A_{mn} = & \frac{8}{9\sqrt{\pi}} Z(kl_{ei})^2 \int_0^\infty dx \frac{x^3}{H_1} e^{-x} L_m^{(1/2)}(x) L_n^{(1/2)}(x) \\ & + \; 3 \int_0^\infty dx e^{-x} \gamma(3/2, x) L_{m-1}^{(3/2)}(x) L_{n-1}^{(3/2)}(x) \\ & - \; 2 \left[\frac{1}{n-1} \int_0^\infty dx x^{5/2} e^{-2x} L_{n-2}^{(5/2)}(x) L_{m-1}^{(3/2)}(x) + (m \leftrightarrow n) \right], \end{aligned} \tag{C.2}$$

and the source term b_m^A (4.112)

$$b_m^A = \int_0^\infty dx x^{1/2} e^{-x} L_m^{(1/2)}(x) S_A, \tag{C.3}$$

Next the linear system

$$A_{mn} c_n^A = b_m^A \tag{C.4}$$

is solved for the unknown expansion coefficients c_n^A (C.1), for each of the three different sources, $A = N, T, R$ (4.93). Finally, the moments J_B^A (4.97) necessary for evaluation of the transport coefficients (4.107, 4.109) are evaluated from the c_n^A, and the values of the transport coefficients normalized to their classical values (where they exist) are output in the order σ/σ_0, α/α_0, χ/χ_0, β_j, β_q, and β_r. We now examine how these steps are performed numerically.

C.1.2 Evaluation of the matrix elements

The diagonal and upper triangle of matrix A (C.2) are stored in the array `A(i,j)` $i \leq j$, since from (C.2) it is clearly symmetric. In order to evaluate these matrix elements, three functions `assolag1(x,m)`, `assolag3(x,m)` and `assolag5(x,m)` are defined that return the value of the associated Laguerre polynomials $L_{m-1}^{(1/2)}(x)$, $L_{m-1}^{(3/2)}(x)$, and $L_{m-1}^{(5/2)}(x)$, for argument x and the order $m-1$. These functions use the arrays `coeff1`, `coeff3`, and `coeff5` that contain the necessary coefficients and are initially read in by the main program from the external files `c1`, `c3`, and `c5`. The generalized incomplete gamma function $\gamma(3/2, x)$ in (C.2) is evaluated by the subroutine `gami3h(x)` [79].

The integrations required in (C.2) are performed by Gaussian integration

$$\int_0^\infty dx\, x^\alpha e^{-x} f(x) = \sum_i^n w_i f(x_i). \tag{C.5}$$

to take advantage of the special form of the integrands (C.5). The abscissa x_i and weights w_i for the n-point integration are calculated in the code by the subroutine `gaulag(x,w,n,`α`)` [79]. The source terms b_m^A in (C.3, 4.112) are stored in the arrays `SN(m+1)`, `ST(m+1)`, and `SR(m+1)`. The first two may be evaluated analytically,

$$\texttt{SN(m)} = \frac{\sqrt{\pi}}{2}\delta_{m1}, \quad \texttt{ST(m)} = -\frac{\sqrt{\pi}}{2}\delta_{m2}, \tag{C.6}$$

but `SR(i)` requires integrations to be performed

$$\texttt{SR(m)} = \frac{2}{3}\int_0^\infty dx\, \frac{x^{3/2}e^{-x}}{H_1} L_{m-1}^{(1/2)}(x). \tag{C.7}$$

C.1.3 Solution to the linear systems

The matrix A_{mn} (C.2) must be inverted in order to obtain the expansion coefficients c_n^A corresponding to the three different source terms, b_n^A (C.3). Cholesky decomposition

factorizes a symmetric positive definite matrix into

$$A = LL^T, \tag{C.8}$$

where L is a lower triangular matrix [79]. The Cholesky decomposition is performed by `choldc(A,npoly,nphys,p)` [79]. The matrix L is returned in the unused lower triangle of `A(i,j)`, and the diagonal in the vector `p(i)`. This needs only to be done once and can be used to solve the three equations with the different sources `SN`, `ST`, and `SR` with no new calculations. The three equations (C.4) are solved by back substitution

$$Ly^A = b^A, \quad L^T c^A = y^A, \tag{C.9}$$

using three calls to the subroutine `cholsl(A,npoly,nphys,p,SA,psiA)` [79], for A=N, R, and T. The expansion coefficients c_m^A are returned in the arrays `psiN(m+1)`, `psiT(m+1)`, and `psiR(m+1)`.

C.1.4 Transport coefficients

Once the expansion coefficients (C.1) are known, the moments J_B^A (4.97), may be determined. The transport coefficients are constructed from the J_B^A moments according to (4.107-4.109), which is simplified as the moments J_A^B are symmetric under interchange of the indices [8],

$$J_B^A = J_A^B = \frac{2}{\sqrt{\pi}} \int_0^\infty dx\, x^{1/2} e^{-x} S_B \psi^A. \tag{C.10}$$

Writing the sources S_A necessary for (C.10) as

$$S_N = L_0^{(1/2)}, \quad S_T = -\frac{2}{3} L_1^{(1/2)} \tag{C.11}$$

using the expansion of the distribution function (4.110, C.1) and the orthogonality properties of the associated Laguerre polynomials [81]

$$\int_0^\infty dx\, x^{1/2} e^{-x} L_n^{(1/2)}(x) L_m^{(1/2)}(x) = \frac{\Gamma(n+3/2)}{n!}\, \delta_{mn}, \tag{C.12}$$

gives the simple expressions for the moments (C.10) in terms of the expansion coefficients

$$\texttt{jNN} = \texttt{psiN}(1), \quad \texttt{jNT} = -\texttt{psiN}(1), \quad \texttt{jNR} = \texttt{psiR}(1), \tag{C.13}$$

$$\texttt{jTN} = \texttt{jNT}, \quad \texttt{jTT} = -\texttt{psiT}(2), \quad \texttt{jTR} = -\texttt{psiR}(2), \tag{C.14}$$

$$\texttt{jRN} = \texttt{jNR}, \quad \texttt{jRT} = \texttt{jTR}, \quad \texttt{jRR} = \frac{2}{\sqrt{\pi}} \sum_{\texttt{i}=1}^{\texttt{npoly}} \texttt{psiR(i)SR(i)}. \tag{C.15}$$

The above jAB moments are related to the J_B^A (C.10) by

$$J_B^A = \frac{Zkl_{ei}}{kv_{Te}} \texttt{jAB} \tag{C.16}$$

because of the form taken for the expansion (C.1). The transport coefficients can then be evaluated by using the definitions (4.107, 4.109). The results are shown in Figure 3.1.1 and are consistent with those in the paper of Bychenkov *et al.* [8].

C.2 Background heat flux

The code `gradient.f` is also able to evaluate the new coefficients related to the heat carrying background state described in Chapter 6. The input is the same for `nonlocal.f` and the output from the program lists the new coefficients in the order $\beta_j^\nabla(Z, kl_{ei})$, $\beta_q^\nabla(Z, kl_{ei})$, and $\beta_r^\nabla(Z, kl_{ei})$ after the nonlocal transport coefficients of Section C.1.1.

The matrix A (C.2) remains the same, so there is no new matrix to invert. The only changes in the program are the introduction of one more equation in (C.4) for ψ^∇ (6.13) (so that A=N, T, R, ∇). The extra source in (C.3) given by

$$S_\nabla = \frac{4}{3\sqrt{\pi}} \left[x^{3/2}(4-x) - \frac{2}{3H_1} x^{3/2} \left(6 - \frac{13}{2}x + x^2 \right) \right]. \tag{C.17}$$

The vector b_m^∇ is stored in the array `snabla(m+1)`

$$\texttt{snabla(m)} = \int_0^\infty dx\, x^{1/2} e^{-x} S_\nabla L_{m-1}^{(1/2)}(x). \tag{C.18}$$

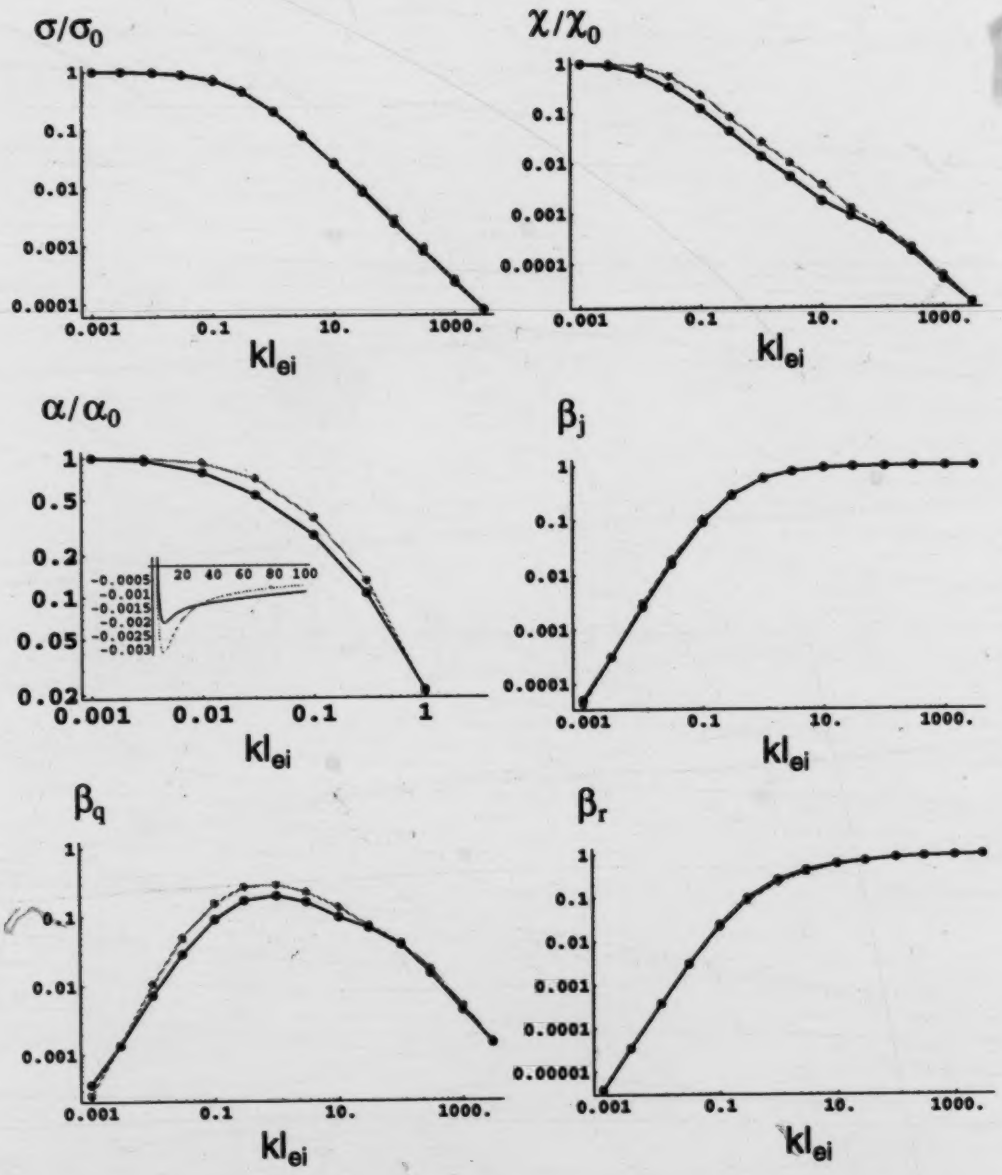

Figure 3.1.1: Plot of the transport coefficients σ, χ, α, β_j, β_q, and β_r calculated by the code `nonlocal.f` for ionization $Z = 8$ (grey lines) and $Z = 64$ (black lines).

On solving the extra equation (one call to `cholsl`) The expansion coefficients for ψ^{∇} are stored in the array `psinabla(m)`, and the new J_B^A moments

$$\texttt{jNnabla} = \texttt{psinabla(1)}, \quad \texttt{jTnabla} = -\texttt{psinabla(2)}, \tag{C.19}$$

and

$$\texttt{jRnabla} = \frac{2}{\sqrt{\pi}} \sum_{i}^{\texttt{npoly}} \texttt{psinabla(i)SR(i)} \tag{C.20}$$

are sufficient to calculate the new coefficients β_j^{∇}, β_q^{∇}, and β_r^{∇}, using the definitions (6.20, 6.21) from Chapter 6.

C.3 Listing of gradient.f

```
c*********************************************************************
c*                                                                   *
c*     This program "gradient.f" solves for the transport            *
c*     coefficients from "Nonlocal Electron Transport in             *
c*     a plasma", PRL 75 4405 (1995) in addition to the              *
c*     new coefficients arising from the presence of                 *
c*                    background gradients                           *
c*                                                                   *
c*   External files required are:                                    *
c*                                                                   *
c*            c1 ---------> coefficients of L_m^(1/2)                *
c*            c3 ---------> coefficients of L_m^(3/2)                *
c*            c5 ---------> coefficients of L_m^(5/2)                *
c*                                                                   *
c*                      Jason Myatt 26/5/97                          *
c*                                                                   *
c*    email: myatt@phys.ualberta.ca                                  *
c*                                                                   *
c*********************************************************************
c
      INTEGER i,j,k,l,n,nmax,nphys
      REAL sqrpi,useful,IR,Inabla
      PARAMETER (sqrpi=1.772453851)
c
c   nmax    -  Maximum number of Gaussian integration points
c   n       -  the number of Gaussian integration points
c   nphys   -  The physical size of the matrices and vectors
c              (determines the maximum number of polynomials)
c   npoly   -  The actual size of the matrices and vectors
c
      PARAMETER (nmax=100,nphys=60)
c
c   Weights, abscissa and alf for Gaussian integration
c   \int_0^\infty{x^alp exp(-x)}=\sum_{i=1}^n{w(j)f[x(j)]}
c
      REAL w(nmax),x(nmax),alf
c
c   Make space for matrix elements
c
      REAL D(nphys,nphys),C1(nphys,nphys),C2(nphys,nphys)
      REAL C3(nphys,nphys),A(nphys,nphys)
      REAL SN(nphys),ST(nphys),SR(nphys),Snabla(nphys)
c
```

```
c   p is the diagonal of the Cholesky decomposition of A,
c   psi(A=N,R,T,nabla) are the sols to the linear system A.psiA=SA,
c   jABs and dABCDs are moments of psiAs occuring in transport
c   coefficients
c
      REAL p(nphys),psiN(nphys),psiT(nphys),psiR(nphys)
      REAL psinabla(nphys)
      REAL jNN,jNT,jNR,jTN,jTT,jTR,jRN,jRT,jRR
      REAL jNnabla,jTnabla,jRnabla
      REAL dNTNT,dRTNT,dNRNT,dNTRT,dNTNR
      REAL dNTnablaT,dNTnablaN
c
c   Transport coefficients
c
      REAL sigma,alpha,chi,betaJ,betaQ,betaR
      REAL betaJnabla,betaQnabla,betaRnabla
c
c   functions and subroutines to be supplied
c
      REAL assolag1,assolag3,assolag5
      REAL SRintegrand,Snablaint,convective,gami3h
c
c  Declare Laguerre "look up" coefficients to be common
c
      REAL coeffs1(36,36),coeffs3(36,36)
      REAL coeffs5(36,36)
c
      COMMON /first/ coeffs1
      COMMON /second/ coeffs3
      COMMON /third/ coeffs5
c
c  The physical parameters needed as input
c
      REAL klei
      INTEGER Z,npoly
c
c  Read in the Laguerre coefficients from external files c1, c3, c5
c
      OPEN(unit=10,file="/Users/myatt/input/c1", status="old")
      READ(10,*) coeffs1
      OPEN(unit=11,file="/Users/myatt/input/c3", status="old")
      READ(11,*) coeffs3
      OPEN(unit=12,file="/Users/myatt/input/c5",status="old")
      READ(12,*) coeffs5
c
```

```
c      Read in  collisionality parameter klei, ionization Z
c      and the number of polynomials npoly.
c
       read*,npoly,klei,Z
c
c-----------------------------------------------------------------
c                     Calculate the matrix, A
c                     ^^^^^^^^^^^^^^^^^^^^^^^
c
c  Obtain the abcissa and weights for n-point integration
c
       alf=0.5
       n=25
c
       CALL gaulag(x,w,n,alf)
c
c
c      Calculation of the Landau damping part of the matrix
c      ^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^
c
c  Note that since D(i,j) is symmetric only the upper triangle and
c  the diagonal will be defined here
c
c     Initialize D(i,j) to zero everywhere
c
       do 45 i=1,nphys
          do 40 j=1,nphys
             D(i,j)=0.0
40        continue
45     continue
c
c     Now evaluate the upper triangle of D(i,j)
c
       do 110 k=1,npoly
          do 105 l=1,k
c                                carry out the integration
             do 100 j=1,n
                D(k,l)=D(k,l)+w(j)*assolag1(x(j),k)*
     *            assolag1(x(j),l)*convective(x(j),klei,Z)
100          continue
c
105       continue
110    continue
c
c
```

```
c
c     Calculation of collision integral part of the matrix
c     ^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^
c
c  Here we calculate the collision integral part of the matrix.
c  This is done in three parts.
c
c
c  We will recalculate the Gaussian integration abcissa and weights
c
c
       alf=0.0
       CALL gaulag(x,w,n,alf)
c
c
c     Initialize C*(i,j)'s to zero everywhere
c
       do 345 i=1,nphys
          do 340 j=1,nphys
             C1(i,j)=0.0
             C2(i,j)=0.0
             C3(i,j)=0.0
340       continue
345    continue
c
c     Now evaluate the upper triangle of C1(i,j)
c                                        ^^^^^^^
       do 410 k=3,npoly
          do 405 l=3,k
c                  carry out the integration
             do 400 j=1,n
                C1(k,l)=C1(k,l)+w(j)*assolag3(x(j),k-1)*
     *            assolag3(x(j),l-1)*gami3h(x(j))
400          continue
c
405       continue
410    continue
c
c
c    Recalculate the Gaussian integration abcissa and weights
c
       alf=2.5
       n=20
       CALL gaulag(x,w,n,alf)
c
```

```
c
c     Now evaluate the upper triangle of C2(i,j)
c
c
       do 510 k=3,npoly
          do 505 l=3,k
c                  carry out the integration
             do 500 j=1,n
                C2(k,l)=C2(k,l)+w(j)*assolag3(x(j),l-1)*
     *            assolag5(x(j),k-2)*exp(-x(j))
500          continue
          C2(k,l)=C2(k,l)/(float(k)-2.0)
c
505       continue
510    continue
c
c     Now evaluate the upper triangle of C3(i,j)
c
c
       do 560 k=3,npoly
          do 555 l=3,k
c                  carry out the integration
             do 550 j=1,n
                C3(k,l)=C3(k,l)+w(j)*assolag3(x(j),k-1)*
     *            assolag5(x(j),l-2)*exp(-x(j))
550          continue
          C3(k,l)=C3(k,l)/(float(l)-2.0)
c
555       continue
560    continue
c
c
c              Now put the whole matrix together:
c
c
c  (N.B. the matrix solver requires A to be upper triangular)
c
       do 680 i=1,npoly
          do 690 j=1,i
             A(j,i)=D(i,j)+3.0*C1(i,j)-2.0*(C2(i,j)+C3(i,j))
690       continue
680    continue
c
c
c            Calculation of  vector sources SA(k)
```

```
c              ---------------------------------------
c
c
c                        Initialize SA(k)
       do 5 k=1,nphys
          SN(k)=0.0
          ST(k)=0.0
          SR(k)=0.0
          Snabla(k)=0.0
5      continue
c
       SN(1)=sqrpi/2.0
       ST(2)=-sqrpi/2.0
c
c                 Carry out the integration for SR
c
       alf=0.5
       n=25
c
       CALL gaulag(x,w,n,alf)
c
       do 20 k=1,npoly
          do 10 j=1,n
             SR(k)=SR(k)+w(j)*SRintegrand(klei,x(j))*
     *         assolag1(x(j),k)
10        continue
20     continue
c
c                 Carry out the integration for Snabla
c
       do 30 k=1,npoly
          do 25 j=1,n
             Snabla(k)=Snabla(k)+w(j)*Snablaint(klei,x(j))*
     *         assolag1(x(j),k)
25        continue
30     continue
c-----------------------------------------------------------------------
c
c                 Solve the linear systems A.psiA=SA
c                 ----------------------------------
c
c   First step is to obtain the Cholesky decompostion of A
c
       CALL choldc(A,npoly,nphys,p)
c
c   Now solve A.x=SA by back substitution
```

```
c
      CALL cholsl(A,npoly,nphys,p,SN,psiN)
      CALL cholsl(A,npoly,nphys,p,ST,psiT)
      CALL cholsl(A,npoly,nphys,p,SR,psiR)
      CALL cholsl(A,npoly,nphys,p,Snabla,psinabla)
c
c  jAB moments are easily obtained from the solution vectors
c
      jNN=psiN(1)
      jNT=-psiN(2)
      jNR=psiR(1)
c
      jTN=jNT
      jTT=-psiT(2)
      jTR=-psiR(2)
c
      jRN=jNR
      jRT=jTR
      jRR=0.0
      do 901 i=1,npoly
         jRR=jRR+psiR(i)*SR(i)
 901  continue
      jRR=(2.0/sqrpi)*jRR
c
c  gradient moments
c
      jNnabla=psinabla(1)
      jTnabla=-psinabla(2)
      jRnabla=0.0
      do 902 i=1,npoly
         jRnabla=jRnabla+psinabla(i)*SR(i)
 902  continue
      jRnabla=(2.0/sqrpi)*jRnabla
c
c  dABCD moments are combinations of the jABs
c
      dNTNT=jNN*jTT-jTN*jNT
      dRTNT=jRN*jTT-jTN*jRT
      dNRNT=jNN*jRT-jRN*jNT
      dNTRT=jNR*jTT-jTR*jNT
      dNTNR=jNN*jTR-jTN*jNR
c
c  gradient moments
c
      dNTnablaT=jNnabla*jTT-jNT*jTnabla
```

```
      dNTnablaN=jNnabla*jTN-jNN*jTnabla
c
c  Evaluate the nonlocal transport coefficients, normalized to
c  classical values whenever they exist.
c
      useful=3.141592654/(Z*(klei**2))
c
      sigma=(3.0/32.0)*useful*(jTT/dNTNT)
      alpha=-(1.0/16.0)*useful*(jNT+jTT)/dNTNT
      chi=(3.0/200.0)*useful*(2.0*jNT+jTT+jNN)/dNTNT
c
      betaJ=1.0-dRTNT/dNTNT
      betaQ=(dRTNT-dNRNT)/dNTNT
c
c  An integration is necessary for betaR
c
      alf=0.0
      n=25
      call gaulag(x,w,n,alf)
c
      IR=0.0
      do 909 i=1,n
         IR=IR+w(i)*(1.0/hone(klei,x(i)))
 909  continue
c
      betaR=1.0+float(Z)*(klei**2.0)*(jRR-jRN*(1.0-betaJ)
     *      -jRT*(1.0-betaJ-betaQ))-IR
c
c  Gradient dependent transport coefficients
c
      betaJnabla=dNTnablaT/dNTNT
      betaQnabla=(dNTnablaT+dNTnablaN)/dNTNT
c
c  An integration is neccessary for betaRnabla
c
      alf=0.5
      n=25
      call gaulag(x,w,n,alf)
c
      Inabla=0.0
      do 910 i=1,n
         Inabla=Inabla+w(i)*(6.0-6.5*x(i)+x(i)**2.0)/hone(klei,x(i))
 910  continue
      Inabla=Inabla*4.0/3.0/sqrpi
c
```

```
      betaRnabla=Inabla-float(Z)*(klei**2.0)*(jRnabla-
     *   jRN*dNTnablaT/dNTNT+jRT*dNTnablaN/dNTNT)
c
      print*,sigma,alpha,chi,betaJ,betaQ,betaR,
     *        betaJnabla,betaQnabla,betaRnabla
c
      END
c
c*********************************************************************
c                    FUNCTIONS and SUBROUTINES
c*********************************************************************
c
  These FUNCTIONs will generate the Associated Laguerre Polynomials
c               L_m^n(x), where n is as follows:
c
c                    assolag1(x)----------> n=1/2
c                    assolag3(x)----------> n=3/2
c                    assolag5(x)----------> n=5/2
c
c   Give the degree of the polynomial the subroutine will return
c   the value for a given x in assolag{1,3,5}. The arrays of
c   coefficients are COMMON and must be defined in the driver
c   program gradient.f
c
c Maximum size of arrays determined by external files c1, c3, c5
c
c
      REAL FUNCTION  assolag1(x,order)
c
c
      REAL coeffs1(36,36),x
      INTEGER order
      COMMON /first/ coeffs1
c
c
      assolag1=coeffs1(order,order)
c
      do 10 j=order-1,1,-1
           assolag1=assolag1*x+coeffs1(j,order)
10    continue
c
c
      return
      END
c
```

```
c-----------------------------------------------------------------
c
      REAL FUNCTION  assolag3(x,order)
c
c
      REAL coeffs3(36,36),x
      INTEGER order
      COMMON /second/ coeffs3
c
c
      assolag3=coeffs3(order,order)
c
      do 10 j=order-1,1,-1
            assolag3=assolag3*x+coeffs3(j,order)
10    continue
c
c
      return
      END
c
c-----------------------------------------------------------------
c
      REAL FUNCTION  assolag5(x,order)
c
c
      REAL coeffs5(36,36),x
      INTEGER order
      COMMON /third/ coeffs5
c
c
      assolag5=coeffs5(order,order)
c
      do 10 j=order-1,1,-1
            assolag5=assolag5*x+coeffs5(j,order)
10    continue
c
c
      return
      END
c
c-----------------------------------------------------------------
c
      REAL FUNCTION hone(klei,x)
c
      REAL klei,x,pi
```

```
      PARAMETER (pi=3.141592654)
c
      hone=Sqrt(1.0+8.0*pi/81.0*(klei**2)*(x**4))
c
      return
      END
c-----------------------------------------------------------------
c
      REAL FUNCTION SRintegrand(klei,x)
c
      REAL klei,x,pi,use,hone
      PARAMETER (pi=3.141592654,use=0.501502)
c
c
      hone=Sqrt(1.0+8.0*pi/81.0*(klei**2)*(x**4))
c
      SRintegrand=(2.0*x)/(3.0*hone)
c
      return
      END
c-----------------------------------------------------------------
c
      REAL FUNCTION Snablaint(klei,x)
c
      REAL klei,x,pi,u,hone
      PARAMETER (pi=3.141592654,u=0.752252778)
c
c
      hone=Sqrt(1.0+8.0*pi/81.0*(klei**2)*(x**4))
c
      Snablaint=u*(4.0-x+2.0*(x**2-6.5*x+6.0)/(3.0*hone))*(x**1.5)
c
      return
      END
```